



Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE 2009

MARKING SCHEME

MATHEMATICS

HIGHER LEVEL

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GENERAL GUIDELINES FOR EXAMINERS – PAPER 1

1. Penalties of three types are applied to candidates' work as follows:

- Blunders - mathematical errors/omissions (-3)
- Slips - numerical errors (-1)
- Misreadings (provided task is not oversimplified) (-1).

Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,..., S1, S2,..., M1, M2,...etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that

- any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
- if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
- a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,...etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.

QUESTION 1

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 (5, 5) marks** **Att (2, 2)**

1. (a) Find the value of $\frac{x}{y}$ when $\frac{2x+3y}{x+6y} = \frac{4}{5}$.

Cross Multiplication **5 marks** **Att 2**
Finish **5 marks** **Att 2**

1 (a)

$$\frac{2x+3y}{x+6y} = \frac{4}{5} \Rightarrow 10x+15y = 4x+24y \Rightarrow 6x = 9y. \quad \therefore \frac{x}{y} = \frac{9}{6} = \frac{3}{2}.$$

Blunders (-3)

B1 Incorrect cross multiplication

Slips (-1)

S1 Numerical

S2 $\frac{y}{x}$

OR

Correct Ratio **5 marks** **Att 2**
Solving **5 marks** **Att 2**

1 (a)

Let numerator = 4 and denominator = 5 (or 8 & 10 respectively, etc.)

$$\Rightarrow (i) : 2x+3y = 4 \quad \times 2 \Rightarrow 4x+6y = 8$$

$$(ii) : x+6y = 5 \quad \times 1 \Rightarrow \underline{x+6y = 5}$$

$$3x = 3 \quad \Rightarrow \quad x = 1$$

(ii): $x+6y = 5$
 (1)+6y = 5

$$6y = 4 \Rightarrow y = \frac{4}{6} = \frac{2}{3}$$

$$\frac{x}{y} = \frac{1}{\left(\frac{2}{3}\right)} = \frac{3}{2}$$

Blunders (-3)

B1 Error in ratio

B2 No $\frac{x}{y}$

Slips (-1)

S1 Numerical

S2 $\frac{y}{x}$

Part (b)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(b) Let $f(x) = x^2 - 7x + 12$.

(i) Show that if $f(x+1) \neq 0$, then $\frac{f(x)}{f(x+1)}$ simplifies to $\frac{x-4}{x-2}$.

(ii) Find the range of values of x for which $\frac{f(x)}{f(x+1)} > 3$.

(b) (i) $f(x+1)$

5 marks

Att 2

Simplification

5 marks

Att 2

1 (b) (i) $f(x) = x^2 - 7x + 12 \Rightarrow f(x+1) = (x+1)^2 - 7(x+1) + 12$.
 $\frac{f(x)}{f(x+1)} = \frac{x^2 - 7x + 12}{x^2 - 5x + 6} = \frac{(x-3)(x-4)}{(x-3)(x-2)} = \frac{x-4}{x-2}$.

Blunders (-3)

B1 Expansion $(x+1)^2$ once only

B2 Incorrect fraction

B3 Factors

(b) (ii) Quadratic Inequality

5 marks

Att 2

Range

5 marks

Att 2

1 (b) (ii)

$$\frac{f(x)}{f(x+1)} > 3 \Rightarrow \frac{x-4}{x-2} > 3$$

Multiply across by $(x-2)^2 > 0$

$$(x-2)(x-4) > 3(x-2)^2$$

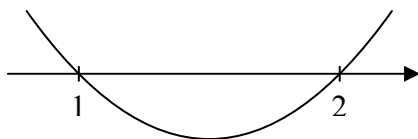
$$x^2 - 6x + 8 > 3(x^2 - 4x + 4)$$

$$x^2 - 6x + 8 > 3x^2 - 12x + 12$$

$$0 > 2x^2 - 6x + 4$$

$$0 > x^2 - 3x + 2$$

$$0 > (x-1)(x-2)$$



Range : $1 < x < 2$

Blunders (-3)

B1 Inequality sign

B2 Indices

B3 Expansion of $(x-2)^2$ once only

B4 Factors

B5 Roots formula once only

- B6 Deduction root from factor
- B7 Range not stated
- B8 Incorrect range
- B9 Shape graph

Slips (-1)

- S1 Numerical

Attempts

- A1 Linear inequality only

Worthless

- W1 Squares both sides

OR
(When not treated as a quadratic)

- | | | |
|--|----------------|--------------|
| (b) (ii) case $(x - 2) > 0$ | 5 marks | Att 2 |
| case $(x - 2) < 0$ | 5 marks | Att 2 |

1 (b) (ii)

case (a): $x - 2 > 0$ (so $x > 2$)

$$\frac{x - 4}{x - 2} > 3$$

$$\Leftrightarrow (x - 4) > 3(x - 2) \quad \text{since } x - 2 > 0$$

$$\Leftrightarrow x - 4 > 3x - 6$$

$$\Leftrightarrow 2 > 2x$$

$$\Leftrightarrow 1 > x$$

Not possible when $x > 2 \Rightarrow$ no solution from this case.

case (b): $x - 2 < 0$ (so $x < 2$)

$$\frac{x - 4}{x - 2} > 3$$

$$\Leftrightarrow x - 4 < 3(x - 2) \quad \text{since } x - 2 < 0$$

$$\Leftrightarrow x - 4 < 3x - 6$$

$$\Leftrightarrow 2 < 2x$$

$$\Leftrightarrow 1 < x$$

$\Rightarrow 1 < x < 2$

OR

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) Given that $x - c + 1$ is a factor of $x^2 - 5x + 5cx - 6b^2$, express c in terms of b .

Division

5 marks

Att 2

Remainder = 0

5 marks

Att 2

Quadratic in b and c

5 marks

Att 2

Values of c

5 marks

Att 2

1 (c)

$$\begin{array}{r} x + (-6 + 6c) \\ x - c + 1 \overline{) x^2 - 5x + 5cx - 6b^2} \\ \underline{x^2 + x - cx} \\ x(-6 + 6c) - 6b^2 \\ \underline{x(-6 + 6c) - c(-6 + 6c) + (-6 + 6c)} \\ -6b^2 + c(-6 + 6c) - (-6 + 6c) \end{array}$$

$$\therefore -6b^2 - 6c + 6c^2 + 6 - 6c = 0$$

$$c^2 - 2c + 1 = b^2.$$

$$(c-1)^2 = b^2 \Rightarrow c-1 = \pm b \Rightarrow c = 1 \pm b.$$

Blunders (-3)

B1 Indices

B2 Not like to like when equation coefficients

B3 Only one value of c given

B4 Factors

Slips (-1)

S1 Not changing sign when subtracting

Attempts

A1 Any effort at division

Other linear factor	5 marks	Att 2
Equating coefficients	5 marks	Att 2
Quadratic in b and c	5 marks	Att 2
Values of c	5 marks	Att 2

1 (c)

$$f(x) = x^2 - 5x + 5cx - 6b^2 = (x - c + 1) \left(x - \frac{6b^2}{-c + 1} \right)$$

$$(x + 1 - c) \left(x - \frac{6b^2}{1 - c} \right) = x^2 - cx + x - \frac{6b^2x}{1 - c} + \frac{6b^2c}{1 - c} - \frac{6b^2}{1 - c}$$

$$= x^2 - x \left(c - 1 + \frac{6b^2}{1 - c} \right) + \frac{6b^2c - 6b^2}{1 - c}$$

Equating Coefficients of x :

$$5 - 5c = c - 1 + \frac{6b^2}{1 - c}$$

$$6 - 6c = \frac{6b^2}{1 - c}$$

$$(1 - c) = \frac{b^2}{(1 - c)}$$

$$(1 - c)^2 = b^2$$

$$1 - c = \pm b$$

$$c = 1 \pm b$$

Blunders (-3)

- B1 Indices
- B2 Only 1 value of c given
- B3 Factors

Root $(c-1)$	5 marks	Att 2
$f(c-1)$ substituted	5 marks	Att 2
Quadratic in b and c	5 marks	Att 2
Values of c	5 marks	Att 2

1 (c)

$$(x - c + 1) \text{ is a factor of } f(x) \Rightarrow (c - 1) \text{ is a root}$$

$$\Rightarrow f(c - 1) = 0$$

$$f(x) = x^2 - 5x + 5cx - 6b^2$$

$$f(c - 1) = (c - 1)^2 - 5(c - 1) + 5c(c - 1) - 6b^2 = 0$$

$$c^2 - 2c + 1 - 5c + 5 + 5c^2 - 5c = 6b^2$$

$$6c^2 - 12c + 6 = 6b^2$$

$$6(c^2 - 2c + 1) = 6(b^2)$$

$$(c - 1)^2 = b^2$$

$$c - 1 = \pm b$$

$$c = 1 \pm b$$

Blunders (-3)

B1 Indices

B2 Expansion of $(c - 1)^2$ once only

B3 Only 1 value of c given

B4 Factors

QUESTION 2

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (10, 5, 5) marks	Att (3, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 (5, 5) marks** **Att (2, 2)**

2. (a) Solve the simultaneous equations

$$x - y + 8 = 0$$

$$x^2 + xy + 8 = 0.$$

Quadratic **5 marks** **Att 2**
Values **5 marks** **Att 2**

2 (a) $x = y - 8. \therefore (y - 8)^2 + y(y - 8) + 8 = 0.$
 $y^2 - 16y + 64 + y^2 - 8y + 8 = 0$
 $2y^2 - 24y + 72 = 0 \Rightarrow y^2 - 12y + 36 = 0.$
 $(y - 6)^2 = 0 \Rightarrow y = 6.$
 \therefore Solution is $(-2, 6).$

Blunders (-3)

- B1 Indices
- B2 Factors once only
- B3 Deduction value from factor
- B4 Not getting 2nd value (having got 1st)
- B5 Roots formula once only

Slips (-1)

- S1 Numerical

Attempts

- A1 Not quadratic

Worthless

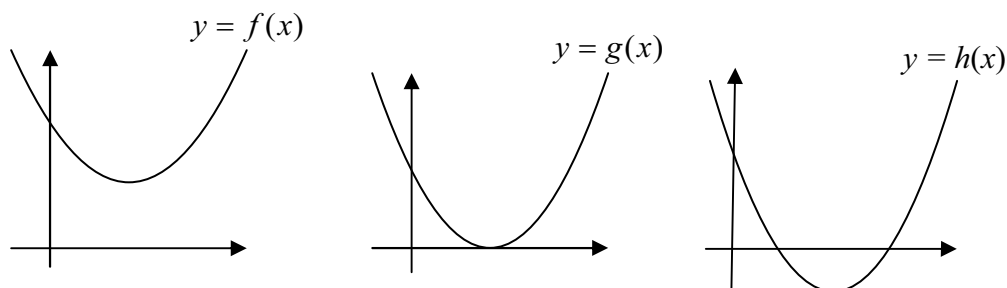
- W1 Trial and error

Part (b)

20 (10, 5, 5) marks

Att (3, 2, 2)

(b) (i) The graphs of three quadratic functions, f , g and h , are shown.



In each case, state the nature of the roots of the function.

(ii) The equation $kx^2 + (1 - k)x + k = 0$ has equal real roots.
Find the possible values of k .

(b) (i)

10 marks

Att 3

2 (b) (i) $f(x)$ has no real roots; (it has two complex roots).
 $g(x)$ has two real equal roots. [or: $g(x)$ has one real root]
 $h(x)$ has two distinct real roots.

Blunders (-3)

B1 Does not state nature of roots, or states incorrect nature of roots.

B2 Does not state number of roots (once only).

Note: One blunder only in each function

(b) (ii) Quadratic

5 marks

Att 2

Values of k

5 marks

Att 2

2 (b) (ii)

$$\text{Equal roots} \Rightarrow b^2 - 4ac = 0.$$

$$\therefore (1 - k)^2 - 4k^2 = 0.$$

$$1 - 2k + k^2 - 4k^2 = 0 \Rightarrow 3k^2 + 2k - 1 = 0.$$

$$(k + 1)(3k - 1) = 0 \Rightarrow k = -1, k = \frac{1}{3}.$$

Blunders (-3)

B1 Indices

B2 Real equal roots condition

B3 Factors once only

B4 Roots formula once only

B5 Deduction of value from factor or no value from factor

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) (i) One of the roots of $px^2 + qx + r = 0$ is n times the other root.
Express r in terms of p, q and n .

(ii) One of the roots of $x^2 + qx + r = 0$ is five times the other.
If q and r are positive integers, determine the set of possible values of q .

(c) (i) Root
Express r

5 marks
5 marks

Att 2
Att 2

2 (c) (i)

Roots are α and $n\alpha$.

$$\therefore \alpha + n\alpha = \frac{-q}{p} \text{ and } \alpha(n\alpha) = \frac{r}{p}.$$

$$\alpha(1+n) = \frac{-q}{p} \Rightarrow \alpha = \frac{-q}{p(1+n)}.$$

$$\text{But } \alpha^2 = \frac{r}{pn} \Rightarrow \frac{q^2}{p^2(1+n)^2} = \frac{r}{pn}.$$

$$\therefore r = \frac{nq^2}{p(n+1)^2}.$$

(c) (ii) r in terms of q
Values of q

5 marks
5 marks

Att 2
Att 2

2 (c) (ii)

$$r = \frac{nq^2}{p(n+1)^2}, \text{ by part (i), where } n = 5 \text{ and } p = 1.$$

$$\therefore r = \frac{5q^2}{36}.$$

For r to be a positive integer, q^2 must be divisible by 36, so q is divisible by 6.

$$\therefore q = \{6, 12, 18, 24, \dots\}.$$

OR

2 (c) (ii) Equation : $x^2 - (-q)x + (r) = 0$

Roots : $\alpha, 5\alpha$

$$x^2 - (\alpha + 5\alpha)x + (5\alpha^2) = 0$$

$$\text{Equating Coefficients: (i) : } 6\alpha = -q \Rightarrow \alpha = -\frac{q}{6}$$

$$\text{(ii) } 5\alpha^2 = r$$

$$5\left(-\frac{q}{6}\right)^2 = r$$

$$r = \frac{5q^2}{36}$$

For r to be a positive integer, q^2 must be divisible by 36, so q is divisible by 6.

$$\therefore q = \{6, 12, 18, 24, \dots\}.$$

Blunders (-3)

- B1 Indices
- B2 Statement quadratic equation once only
- B3 Incorrect sum roots
- B4 Incorrect product roots
- B5 One value of q only or two values q

Slips (-1)

- S1 Numerical

QUESTION 3

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 (5, 5) marks** **Att (2, 2)**

3 (a) $z_1 = a + bi$ and $z_2 = c + di$, where $i^2 = -1$.

Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, where \bar{z} is the complex conjugate of z .

$\overline{z_1 + z_2}$	5 marks	Att 2
$\overline{z_1} + \overline{z_2}$	5 marks	Att 2

3 (a) $\overline{z_1} = a - bi, \overline{z_2} = c - di \Rightarrow \overline{z_1} + \overline{z_2} = (a + c) - (b + d)i.$

$$\overline{z_1 + z_2} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i = \overline{z_1} + \overline{z_2}.$$

Blunders (-3)

B1 i

B2 Conjugate

Part (b)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(b) Let $A = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$.

(i) Express A^3 in the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbf{Z}$.

(ii) Hence, or otherwise, find A^{17} .

(b) (i) A^2
 A^3

5 marks

Att 2

5 marks

Att 2

3 (b) (i)

$$A^2 = \frac{1}{4} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix}$$

$$\therefore A^3 = \frac{1}{8} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) (ii) Values in A^{17}

5 marks

Att 2

A^{17} calculated

5 marks

Att 2

3 (b) (ii)

$$\begin{aligned} A^{17} &= (A^3)^5 A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^5 \frac{1}{4} \begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2\sqrt{3} \\ -2\sqrt{3} & 2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \end{aligned}$$

Blunders (-3)

B1 Indices

Slips (-1)

S1 Numerical

S2 Each incorrect element

Note: Can only get Att 2 in (ii) if A^3 not a diagonal matrix (in second 5 marks).

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) (i) Use De Moivre's theorem to prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(ii) Hence, find $\int \sin^3\theta d\theta$.

(c) (i) Sin 3θ

5 marks

Att 2

Value

5 marks

Att 2

3 (c) (i)

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + 3i\cos^2\theta\sin\theta - i\sin^3\theta.\end{aligned}$$

$$\begin{aligned}\therefore \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

(c) (ii) $\int \sin^3\theta d\theta$

5 marks

Att 2

Finish

5 marks

Att 2

3 (c) (ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$

$$\therefore \int \sin^3\theta d\theta = \frac{1}{4} \int (3\sin\theta - \sin 3\theta) d\theta = \frac{1}{4} \left[-3\cos\theta + \frac{1}{3}\cos 3\theta \right] + C.$$

Note: Not "hence" \Rightarrow zero marks for integration.

Blunders (-3)

B1 Statement De Moivre once only

B2 Binomial expansion once only

B3 i

B4 Indices

B5 Trig formula

B6 Not like to like when equating coefficients

B7 Integration

B8 C omitted

QUESTION 4

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 (5, 5) marks** **Att (2, 2)**

4. (a) Three consecutive terms of an arithmetic series are $4x+11$, $2x+11$, and $3x+17$.
Find the value of x .

Definition of A.P. **5 marks** **Att 2**
Value x **5 marks** **Att 2**

4 (a)

$$(2x+11) - (4x+11) = (3x+17) - (2x+11).$$

$$-2x = x + 6 \Rightarrow x = -2.$$

(And the three terms are 3, 7 and 11.)

Blunders (-3)
B1 AP statement

Slips (-1)
S1 Numerical

Worthless
W1 Geometric sequence
W2 Puts in values for x

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) (i) Show that $\frac{2}{r^2-1} = \frac{1}{r-1} - \frac{1}{r+1}$, where $r \neq \pm 1$.

(ii) Hence, find $\sum_{r=2}^n \frac{2}{r^2-1}$.

(iii) Hence, evaluate $\sum_{r=2}^{\infty} \frac{2}{r^2-1}$.

(b) (i) **5 marks** **Att 2**

4 (b) (i)

$$\frac{1}{r-1} - \frac{1}{r+1} = \frac{r+1-r-1}{(r-1)(r+1)} = \frac{2}{r^2-1}.$$

OR

4 (b) (i)

$$\text{Let } \frac{2}{r^2 - 1} = \frac{a}{r - 1} - \frac{b}{r + 1}$$

$$2 = q(r + 1) - b(r - 1)$$

$$(0)r + (2) = (a - b)r + (a + b)$$

Equating Coefficients : (i) : $a - b = 0$
(ii) : $a + b = 2$

$$(i) : a - b = 0$$

$$(ii) : a + b = 2$$

$$\frac{2a}{2} = 2$$

$$a = 1$$

$$(i) a - b = 0 \Rightarrow a = b \Rightarrow a = b = 1$$

$$\frac{2}{r^2 - 1} = \frac{1}{r - 1} - \frac{1}{r + 1}$$

(b)(ii) Set up cancellation
Finish

5 marks
5 marks

Att 2
Att 2

4 (b) (ii)

$$\sum_{r=2}^n \frac{2}{r^2 - 1} = \sum_{r=2}^n \left(\frac{1}{r - 1} - \frac{1}{r + 1} \right)$$

$$= \sum_{r=2}^n \left(\frac{1}{r - 1} \right) - \sum_{r=2}^n \left(\frac{1}{r + 1} \right)$$

$$= \sum_{r=1}^{n-1} \frac{1}{r} - \sum_{r=3}^{n+1} \frac{1}{r}$$

$$= \left(1 + \frac{1}{2} + \sum_{r=3}^{n-1} \frac{1}{r} \right) - \left(\sum_{r=3}^{n-1} \frac{1}{r} + \frac{1}{n} + \frac{1}{n+1} \right)$$

$$= \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

OR

(b)(ii) Terms U_2 to U_n
Sum to n terms

5 marks
5 marks

Att 2
Att 2

4 (b) (ii)

$$\begin{aligned}U_n &= \frac{1}{n^2 - 1} &= \frac{1}{n-1} - \frac{1}{n+1} \\U_{n-1} & &= \frac{1}{n-2} - \frac{1}{n} \\U_{n-2} & &= \frac{1}{n-3} - \frac{1}{n-1} \\& \vdots & \vdots \\& \vdots & \vdots \\U_4 & &= \frac{1}{3} - \frac{1}{5} \\U_3 & &= \frac{1}{2} - \frac{1}{4} \\U_2 & &= \frac{1}{1} - \frac{1}{3} \\S_n &= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \\S_n &= \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\end{aligned}$$

(b) (iii) Sum to infinity

5 marks

Att 2

4 (b) (iii)

$$\sum_{r=2}^{\infty} \frac{2}{r^2 - 1} = \text{Limit}_{n \rightarrow \infty} \left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1} \right) = \frac{3}{2}$$

Blunders (-3)

- B1 Indices
- B2 Cancellation must be shown or implied
- B3 Not like to like when equating coefficients
- B4 Term omitted
- B5 Gets S_r

Slips (-1)

- S1 Numerical

Note: Must show three terms at start and two terms at finish or *vice versa*.

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) A finite geometric sequence has first term a and common ratio r .
The sequence has $2m + 1$ terms, where $m \in \mathbf{N}$.

(i) Write down the last term, in terms of a , r , and m .

(ii) Write down the middle term, in terms of a , r , and m .

(iii) Show that the product of all of the terms of the sequence is equal to the middle term raised to the power of the number of terms.

Part (c) (i)

5 marks

Att 2

4 (c) (i) Last term = ar^{2m} .

Part (c) (ii)

5 marks

Att 2

4 (c) (ii) Middle term = ar^m .

(c) (iii) Product

5 marks

Att 2

Show

5 marks

Att 2

4 (c) (iii)

Product of terms = $a \times ar \times ar^2 \times \dots \times ar^{2m}$
 $= a^{2m+1} \times r^{0+1+2+\dots+2m}$. [$0 + 1 + 2 + \dots + 2m$ is an A.P. with $2m + 1$ terms]
 $= a^{2m+1} \left(r^{\frac{(2m+1)(2m)}{2}} \right) = a^{2m+1} r^{m(2m+1)}$
 $= (ar^m)^{2m+1}$.

Blunders (-3)

B1 Indices

B2 $U_n \neq AR^{n-1}$

B3 Formula AP

B4 Incorrect substitution into formula once only

B5 Middle term

Slips (-1)

S1 Numerical

QUESTION 5

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (5, 5, 10) marks	Att (2, 2, 3)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 (5, 5)marks** **Att (2, 2)**

5 (a) **(a)** Solve for x : $x - 2 = \sqrt{3x - 2}$.

Quadratic **5 marks** **Att 2**
Solution **5 marks** **Att 2**

5 (a)

$$x - 2 = \sqrt{3x - 2} \Rightarrow (x - 2)^2 = 3x - 2.$$

$$x^2 - 4x + 4 = 3x - 2 \Rightarrow x^2 - 7x + 6 = 0.$$

$$(x - 6)(x - 1) = 0 \Rightarrow x = 6 \text{ and } x = 1.$$

Test: $x = 1$	LHS: $(x - 2) = (1 - 2) = -1$
	RHS: $\sqrt{3x - 2} = \sqrt{1} = 1$
$x \neq 1$	

$x = 6$	LHS: $x - 2 = 6 - 2 = 4$
	RHS: $\sqrt{3x - 2} = \sqrt{16} = 4$

Solution: $x = 6$

Blunders (-3)

- B1 Indices
- B2 Expansion $(x - 2)^2$ once only
- B3 Factors once only
- B4 Roots formula once only
- B5 Deduction value from factor
- B6 Excess value

Slips (-1)

- S1 Numerical

Attempts

- A1 $x = 6$ and no other work merits Att 2
- A2 $x = 6$ by trial and error merits Att 2

(b) Prove by induction that, for all positive integers n , 5 is a factor of $n^5 - n$.

P(1)

5 marks

Att 2

P(k)

5 marks

Att 2

P(k+1)

10 marks

Att 3

5 (b)

Let $P(n)$ be the proposition that 5 is a factor of $n^5 - n$.

Test $P(1)$: $1 - 1 = 0$, which is divisible by 5.

Assume $P(k)$: $k^5 - k$ is divisible by 5.

Try to deduce $P(k+1)$: that $(k+1)^5 - (k+1)$ is divisible by 5.

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1.$$

$$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$$

↓

div. by 5 from P(k)

↘

has 5 as factor

so sum is divisible by 5, given $P(k)$.

We have $P(1)$ and $\{P(k) \Rightarrow P(k+1)\}$. Hence, $P(n)$ for all positive integers n .

OR

5 (b)

To prove: $(n^5 - n)$ is divisible by 5

$n = 1$: $1^5 - 1 = 0$, which is divisible by 5

\Rightarrow true for $n = 1$

Assume true for $n = k$: $k^5 - k$ is divisible by 5.

To prove: $(k+1)^5 - (k+1)$ is divisible by 5.

Let $f(k) = k^5 - k$. Given the assumption that $f(k)$ is divisible by 5, then $f(k+1)$ will be divisible by 5 if and only if $[f(k+1) - f(k)]$ is divisible by 5.

$$\begin{aligned} \text{Now, } f(k+1) - f(k) &= [(k+1)^5 - (k+1)] - [k^5 - k] \\ &= [k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1] - k^5 + k \\ &= 5k^4 + 10k^3 + 10k^2 + 5k \\ &= 5(k^4 + 2k^3 + 2k^2 + k), \text{ which is divisible by 5.} \end{aligned}$$

So, the statement is true for $n = k + 1$ whenever it is true for $n = k$.

Since it is true for $n = 1$, then, by induction, it is true for all positive integers.

Blunders (-3)

B1 Binomial expansion once only

B2 Indices

B3 Expansion of $(k+1)^5$ once only

Note: Must prove $P(1)$ step (not sufficient to state $P(n)$ true for $n = 1$).

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) Solve the simultaneous equations

$$\log_3 x + \log_3 y = 2$$

$$\log_3(2y - 3) - 2\log_9 x = 1.$$

One var. in terms of the other

5 marks

Att 2

Change of base

5 marks

Att 2

Quadratic

5 marks

Att 2

Solution

5 marks

Att 2

5 (c)

$$\log_3 x + \log_3 y = 2$$

$$\log_3(xy) = 2$$

$$xy = 9$$

$$x = \frac{9}{y}$$

$$\log_3(2y - 3) - 2\log_9 x = 1$$

$$\log_3(2y - 3) - 2\frac{\log_3 x}{\log_3 9} = 1$$

$$\log_3(2y - 3) - 2\frac{\log_3 x}{2} = 1$$

$$\log_3\left(\frac{2y - 3}{x}\right) = 1$$

$$\frac{2y - 3}{x} = 3$$

$$(2y - 3)\frac{y}{9} = 3$$

$$2y^2 - 3y - 27 = 0$$

$$(2y - 9)(y + 3) = 0$$

$$y > 0 \Rightarrow y \neq -3, \text{ so } y = \frac{9}{2}, \text{ giving } x = 2.$$

Blunders (-3)

B1 Logs

B2 Indices

B3 Formula change of base

B4 Factors

B5 Roots formula

B6 Deduction root from factor or no deduction

B7 Excess value

Worthless

W1 Drops "logs"

Note Must have a quadratic equation for last 5 marks

QUESTION 6

Part (a)	10 marks	Att 3
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 marks** **Att 3**

6 (a) Differentiate $\sin(3x^2 - x)$ with respect to x .

6 (a)

$$f(x) = \sin(3x^2 - x) \Rightarrow f'(x) = \cos(3x^2 - x)(6x - 1).$$

Blunders (-3)

B1 Differentiation

Attempts

A1 Error in differentiation formula

Part (b) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

(b) (i) Differentiate \sqrt{x} with respect to x , from first principles.

(ii) An object moves in a straight line such that its distance from a fixed point is given by $s = \sqrt{t^2 + 1}$, where s is in metres and t is in seconds. Find the speed of the object when $t = 5$ seconds.

(b)(i) $f(x+h) - f(x)$ **5 marks** **Att 2**
Multiplication **5 marks** **Att 2**
Finish **5 marks** **Att 2**

6 (b) (i)

$$f(x) = \sqrt{x} \Rightarrow f(x+h) = \sqrt{x+h}$$

$$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{x}$$

$$= \frac{(\sqrt{x+h} - \sqrt{x})}{1} \times \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{x+h-x}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{h}{\sqrt{x+h} + \sqrt{x}}$$

$$\therefore \text{Limit}_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \text{Limit}_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

OR

6 (b) (i)

$$\begin{aligned}y &= \sqrt{x} \\y + \Delta y &= \sqrt{x + \Delta x} \\ \Delta y &= \sqrt{x + \Delta x} - \sqrt{x} \\ \frac{\Delta y}{\Delta x} &= \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \frac{(\sqrt{x + \Delta x})^2 - (\sqrt{x})^2}{\Delta x [\sqrt{x + \Delta x} + \sqrt{x}]} \\ &= \frac{x + \Delta x - x}{\Delta x (\sqrt{x + \Delta x} + \sqrt{x})} \\ \frac{\Delta y}{\Delta x} &= \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \\ \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}\end{aligned}$$

Blunders (-3)

- B1 $f(x + h)$ or $(x + \Delta x)$
- B2 Indices
- B3 No limits shown or implied or no indication $h \rightarrow 0$
- B4 $h \rightarrow \infty$
- B5 Conjugate
- B6 No left hand side

Worthless

- W1 Not 1st principles

(b) (ii)

5 marks

Att 2

6 (b) (ii)

$$\begin{aligned}s &= (t^2 + 1)^{\frac{1}{2}} \Rightarrow \frac{ds}{dt} = \frac{1}{2}(t^2 + 1)^{-\frac{1}{2}} \cdot 2t = \frac{t}{\sqrt{t^2 + 1}} \\ \therefore \text{At } t = 5, \frac{ds}{dt} &= \frac{5}{\sqrt{26}} \text{ metres per second.}\end{aligned}$$

Blunders (-3)

- B1 Differentiation
- B2 Indices
- B3 No substitution $t = 5$

Slips (-1)

- S1 Incorrect units or omitted units

Attempts

- A1 Error in differentiation formula

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) The equation of a curve is $y = \frac{2}{x-3}$.

(i) Write down the equations of the asymptotes and hence sketch the curve.

(ii) Prove that no two tangents to the curve are perpendicular to each other.

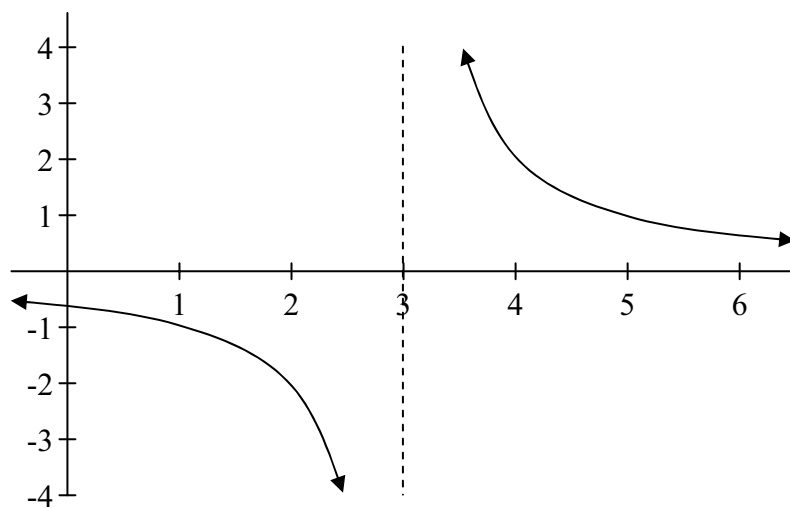
(c) (i) Asymptotes
Sketch

5 marks
5 marks

Att 2
Att 2

6 (c) (i)

Equations of asymptotes are $x = 3$ and $y = 0$.



(c) (ii) Slope
Deduction

5 marks
5 marks

Att 2
Att 2

6 (c) (ii)

$$y = \frac{2}{x-3} = 2(x-3)^{-1} \Rightarrow \frac{dy}{dx} = -2(x-3)^{-2} = \frac{-2}{(x-3)^2}$$

$$\therefore \text{Slope of tangent at } (x, y) \text{ is } m = \frac{-2}{(x-3)^2}$$

But m will be negative for all values of $x \Rightarrow m_1 \cdot m_2 \neq -1$

\therefore No two tangents are perpendicular to each other.

OR

6 (c) (ii)

$$y = 2(x - 3)^{-1}$$

$$m = \frac{dy}{dx} = \frac{-2}{(x - 3)^2}$$

Let tangents at $x = a$ and $x = b$ be perpendicular

$$\text{At } x = a: m_1 = \frac{-2}{(a - 3)^2}$$

$$\text{At } x = b: m_2 = \frac{-2}{(b - 3)^2}$$

$$(m_1)(m_2) = \frac{-2}{(a - 3)^2} \cdot \frac{-2}{(b - 3)^2} = \frac{4}{(a - 3)^2(b - 3)^2} \neq -1, \text{ (since LHS is positive).}$$

\Rightarrow Tangents cannot be perpendicular.

Blunders (-3)

B1 Indices

B2 Asymptote

B3 Differentiation

B4 Slope $\neq \frac{dy}{dx}$

B5 $m_1 m_2 \neq -1$

B6 Incorrect deduction or no deduction

Slips

S1 Curve not approaching asymptotes.

Attempts

A1 Error in differentiation formula

Worthless

W1 Integration

QUESTION 7

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 (5, 5) marks** **Att (2, 2)**

7 (a) **(a)** The equation of a curve is $x^2 - y^2 = 25$. Find $\frac{dy}{dx}$ in terms of x and y .

Differentiate **5 marks** **Att 2**

Isolate $\frac{dy}{dx}$ **5 marks** **Att 2**

7 (a) $x^2 - y^2 = 25 \Rightarrow 2x - 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$.

OR

<p>7 (a) $x^2 - y^2 = 25$ $y^2 = x^2 - 25$ $y = \sqrt{x^2 - 25}$ $y = (x^2 - 25)^{\frac{1}{2}}$ $\frac{dy}{dx} = \frac{1}{2}(x^2 - 25)^{-\frac{1}{2}} \cdot 2x$ $= \frac{x}{\sqrt{x^2 - 25}}$ $= \frac{x}{y}$</p>	<p>OR $y = -\sqrt{x^2 - 25}$ $y = -(x^2 - 25)^{\frac{1}{2}}$ $\frac{dy}{dx} = -\left[\frac{1}{2}(x^2 - 25)^{-\frac{1}{2}} \cdot 2x\right]$ $= -\left[\frac{x}{\sqrt{x^2 - 25}}\right]$ $= \frac{x}{y}$</p>
$\frac{dy}{dx} = \frac{x}{y}$	

Blunders (-3)

B1 Differentiation

B2 Indices

Attempts

A1 Error in differentiation formula

A2 $\frac{dy}{dx} = 2x - 2y \frac{dy}{dx}$ and uses two $\frac{dy}{dx}$ terms in first 5 marks.

Worthless

W1 No differentiation

W2 Integration

Part (b)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(b) A curve is defined by the parametric equations

$$x = \frac{3t}{t^2 - 2} \text{ and } y = \frac{6}{t^2 - 2}, \text{ where } t \neq \pm\sqrt{2}.$$

(i) Find $\frac{dy}{dx}$ in terms of t .

(ii) Find the equation of the tangent to the curve at the point given by $t = 2$.

(b) (i) $\frac{dx}{dt}, \frac{dy}{dt}$

5 marks

Att 2

$\frac{dy}{dx}$

5 marks

Att 2

7 (b) (i)

$$x = \frac{3t}{t^2 - 2} \Rightarrow \frac{dx}{dt} = \frac{3(t^2 - 2) - 3t \cdot 2t}{(t^2 - 2)^2} = \frac{-3t^2 - 6}{(t^2 - 2)^2}.$$

$$y = \frac{6}{t^2 - 2} = 6(t^2 - 2)^{-1} \Rightarrow \frac{dy}{dt} = -6(t^2 - 2)^{-2} \cdot 2t = \frac{-12t}{(t^2 - 2)^2}.$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-12t}{(t^2 - 2)^2} \cdot \frac{(t^2 - 2)^2}{-3t^2 - 6} = \frac{12t}{3t^2 + 6} = \frac{4t}{t^2 + 2}.$$

(b)(ii) Slope, point
Equation

5 marks

Att 2

5 marks

Att 2

7 (b) (ii)

$$t = 2 \Rightarrow x = \frac{6}{2} = 3 \text{ and } t = 2 \Rightarrow y = \frac{6}{2} = 3. \therefore \text{Point is } (3, 3).$$

$$\text{Slope of tangent at } t = 2 \text{ is } \frac{8}{6} = \frac{4}{3}.$$

$$\therefore \text{Equation of tangent: } y - 3 = \frac{4}{3}(x - 3) \Rightarrow 4x - 3y - 3 = 0.$$

Blunders (-3)

B1 Differentiation

B2 Indices

B3 Error in getting $\frac{dy}{dx}$

B4 Equation of tangent

B5 Error in slope formula.

Slips (-1)

S1 Numerical

Attempts

A1 Error in differentiation formula

Part (c)

20 (5, 5, 5, 5)

Att (2, 2, 2, 2)

(c) The function $f(x) = x^3 - 3x^2 + 3x - 4$ has only one real root.

(i) Show that the root lies between 2 and 3.

Anne and Barry are each using the Newton-Raphson method to approximate the root. Anne is starting with 2 as a first approximation and Barry is starting with 3.

(ii) Show that Anne's starting approximation is closer to the root than Barry's. (That is, show that the root is less than 2.5.)

(iii) Show, however, that Barry's next approximation is closer to the root than Anne's.

(c) (i)

5 marks

Att 2

$$\begin{aligned}f(x) &= x^3 - 3x^2 + 3x - 4. \\f(2) &= 8 - 12 + 6 - 4 = -2 < 0. \\f(3) &= 27 - 27 + 9 - 4 = 5 > 0. \\\therefore \text{root lies between 2 and 3.}\end{aligned}$$

(c) (ii)

5 marks

Att 2

$$\begin{aligned}f(2.5) &= (2.5)^3 - 3(2.5)^2 + 3(2.5) - 4 \\&= 15.625 - 18.75 + 7.5 - 4 \\&= 0.375 \\f(2) < 0 \text{ and } f(2.5) > 0. \therefore \text{root is between 2 and 2.5.} \\ \text{So, root is closer to 2 than to 3.}\end{aligned}$$

(c) (iii) **Formula + Differentiation**
Finish

5 marks
5 marks

Att 2
Att 2

7 (c) (iii)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where } f(x) = x^3 - 3x^2 + 3x - 4 \text{ and } f'(x) = 3x^2 - 6x + 3$$

$$\text{Ann: } f(2) = -2 \text{ and } f'(2) = 3. \quad x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-2}{3} = 2\frac{2}{3} = 2.666\dots$$

$$\text{Barry: } f(3) = 5 \text{ and } f'(3) = 12. \quad x_2 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{5}{12} = 2\frac{7}{12} = 2.583\dots$$

Both of these are above the root, so the lower one is closer (i.e. Barry's).

Blunders (-3)

- B1 Indices
- B2 Incorrect deduction from $f(2)$ and $f(3)$ or no deduction
- B3 No $f(2.5)$
- B4 Newton-Raphson formula
- B5 Differentiation
- B6 Incorrect deduction or no deduction from work in (iii)

QUESTION 8

Part (a)	10 marks	Att 3
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 marks** **Att 3**

8 (a) Find $\int \left(6x + 3 + \frac{1}{x^2}\right) dx$.

8 (a)

$$\int \left(6x + 3 + \frac{1}{x^2}\right) dx = 3x^2 + 3x - \frac{1}{x} + C.$$

Blunders (-3)

- B1 Integration
- B2 Indices
- B3 No c

Attempts

- A1 Only c correct

Worthless

- W1 Differentiation for integration

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) Evaluate **(i)** $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 3x \sin x \, dx$ **(ii)** $\int_{\ln 3}^{\ln 8} e^x \sqrt{1 + e^x} \, dx$.

Integration **5 marks** **Att 2**
Value **5 marks** **Att 2**

8 (b) (i)

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 3x \sin x \, dx = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2x - \cos 4x) dx = \frac{1}{2} \left[\frac{1}{2} \sin 2x - \frac{1}{4} \sin 4x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \left[\left(\frac{1}{2} \sin \frac{\pi}{2} - \frac{1}{4} \sin \pi \right) - \left(\frac{1}{2} \sin \left(-\frac{\pi}{2} \right) - \frac{1}{4} \sin(-\pi) \right) \right]$$

$$= \frac{1}{2} \left[\left(\frac{1}{2} - 0 \right) - \left(-\frac{1}{2} - 0 \right) \right] = \frac{1}{2}.$$

Integration
Value

5 marks
5 marks

Att 2
Att 2

8 (b) (ii)

Let $u = 1 + e^x$. $\therefore du = e^x dx$.

$$\int_{\ln 3}^{\ln 8} e^x \sqrt{1 + e^x} dx = \int_{1+e^{\ln 3}}^{1+e^{\ln 8}} u^{\frac{1}{2}} du, \text{ but } e^{\ln 8} = 8 \text{ and } e^{\ln 3} = 3.$$
$$= \int_4^9 u^{\frac{1}{2}} du = \left[\frac{2}{3} u^{\frac{3}{2}} \right]_4^9 = \frac{2}{3} [27 - 8] = \frac{38}{3}.$$

OR

8 (b) (ii)

Using x limits:

$$\int_{\ln 3}^{\ln 8} e^x \sqrt{1 + e^x} dx = \frac{2}{3} u^{\frac{3}{2}} \Big|_{x=\ln 3}^{x=\ln 8}$$
$$= \frac{2}{3} (1 + e^x)^{\frac{3}{2}} \Big|_{\ln 3}^{\ln 8}$$
$$= \frac{2}{3} \left[(1 + e^{\ln 8})^{\frac{3}{2}} - (1 + e^{\ln 3})^{\frac{3}{2}} \right]$$
$$= \frac{2}{3} \left[(9)^{\frac{3}{2}} - (4)^{\frac{3}{2}} \right]$$
$$= \frac{2}{3} [27 - 8] = \frac{2}{3} (19) = \frac{38}{3}$$

OR

8 (b) (ii)

$$\int_{\ln 3}^{\ln 8} e^x \sqrt{1 + e^x} dx$$

Let $u = e^x$

$$= \int \sqrt{1 + e^x} \cdot e^x dx$$
$$= \int (1 + u)^{\frac{1}{2}} du$$
$$= \frac{2}{3} (1 + u)^{\frac{3}{2}} \Big|_{x=\ln 3}^{x=\ln 8}$$
$$= \frac{2}{3} (1 + e^x)^{\frac{3}{2}} \Big|_{\ln 3}^{\ln 8}$$
$$= \frac{2}{3} \left[(1 + e^{\ln 8})^{\frac{3}{2}} - (1 + e^{\ln 3})^{\frac{3}{2}} \right]$$
$$= \frac{2}{3} \left[(1 + 8)^{\frac{3}{2}} - (1 + 3)^{\frac{3}{2}} \right]$$
$$= \frac{2}{3} \left[(9)^{\frac{3}{2}} - (4)^{\frac{3}{2}} \right] = \frac{2}{3} [27 - 8] = \frac{2}{3} (19) = \frac{38}{3}$$

* Incorrect substitution and unable to finish yields attempt at most

Blunders (-3)

- B1 Trig formula
- B2 Integration
- B3 Differentiation
- B4 Limits
- B5 Incorrect order in applying limits
- B6 Not calculating substituted limits
- B7 Not changing limits
- B8 Indices
- B9 Logs
- B10 $e^{\ln a} \neq a$

Slips (-1)

- S1 Numerical
- S2 Trig value
- S3 Answer not tidied up

Worthless

- W1 Differentiation instead of integration except where other work merits attempts

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) Use integration methods to establish the standard formula for the volume of a cone.

Diagram + slope

5 marks

Att 2

Correct subst. into volume formula

5 marks

Att 2

Integration

5 marks

Att 2

Volume

5 marks

Att 2

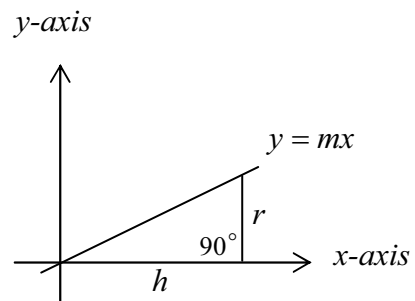
8 (c)

$$y = mx \Rightarrow y = \frac{r}{h}x.$$

Volume of cone = $\pi \int_0^h y^2 dx$, where $y = \frac{r}{h}x$.

$$V = \pi \int_0^h \frac{r^2}{h^2} x^2 dx = \frac{1}{3} \pi \frac{r^2}{h^2} [x^3]_0^h = \frac{1}{3} \pi \frac{r^2}{h^2} h^3$$

$$V = \frac{1}{3} \pi r^2 h.$$



Blunders (-3)

B1 Integration

B2 Slope of line

B3 Equation of line

B4 Volume formula provided it is quadratic

B5 Limits

B6 No Limits

B7 Incorrect order in applying limits

B8 Indices

Slips (-1)

S1 Numerical

Attempts

A1 Uses $v = \pi y$

Worthless

W1 Differentiation instead of integration



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State Examinations Commission

LEAVING CERTIFICATE 2009

MARKING SCHEME

MATHEMATICS - PAPER 2

HIGHER LEVEL

GENERAL GUIDELINES FOR EXAMINERS – PAPER 2

1. Penalties of three types are applied to candidates' work as follows:

- Blunders - mathematical errors/omissions (-3)
- Slips - numerical errors (-1)
- Misreadings (provided task is not oversimplified) (-1).

Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,..., S1, S2,..., M1, M2,...etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that

- any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
- if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
- a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,...etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.

QUESTION 1

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (10, 10) marks	Att (3, 3)
Part (c)	20 (10, 5, 5) marks	Att (3, 2, 2)

Part (a) **10(5, 5) marks** **Att (2, 2)**

1 (a) Show that, for all values of $t \in \mathbf{R}$, the point $\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$ lies on the circle $x^2 + y^2 = 1$.

Part (a) Substitution **5 marks** **Att 2**
Finish **5 marks** **Att 2**

1 (a)

$$x^2 + y^2 = \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2} = \frac{4t^2 + 1 - 2t^2 + t^4}{(1+t^2)^2} = \frac{1 + 2t^2 + t^4}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1.$$

Blunders (-3)

- B1 Incorrect squaring (apply once if same type of error)
- B2 Incorrect factors
- B3 Incorrect conclusion

Slips (-1)

- S1 Arithmetic error

Attempts (2, 2 marks)

- A1 Some correct substitution for x or y
- A2 Effort at expressing t^2 in terms of y

Part (b) **20 (10, 10) marks** **Att (3, 3)**

(b) (i) Find the equation of the tangent to the circle $x^2 + y^2 = 10$ at the point $(3, 1)$.

(ii) Find the values of $k \in \mathbf{R}$ for which the line $x - y + k = 0$ is a tangent to the circle $(x - 3)^2 + (y + 4)^2 = 50$.

Part (b) (i) **10 marks** **Att 3**

1 (b) (i)

Equation of tangent: $xx_1 + yy_1 = r^2 \Rightarrow 3x + y = 10$.

or

Centre of circle $(0,0) \Rightarrow$ Slope diameter $= \frac{1}{3} \Rightarrow$ Slope Tangent $= -3$

Equation of tangent: $y - 1 = -3(x - 3)$

Blunders (-3)

- B1 Error in slope formula
- B2 Slope of tangent not perpendicular to the diameter
- B3 Error in equation of line formula
- B4 Error in equation of tangent formula
- B5 Incorrect centre of circle

Slips (-1)

- S1 Arithmetic error

Attempts (3 marks)

- A1 Equation of tangent formula
- A2 Slope of diameter only
- A3 Equation of line with some substitution

Part (b) (ii)

10 marks

Att 3

1 (b) (ii) Centre $(3, -4)$ and radius $= \sqrt{50} = 5\sqrt{2}$.
Since a tangent, perpendicular distance from centre $(3, -4)$ to $x - y + k = 0$ equals radius.
 $\therefore \left| \frac{3 + 4 + k}{\sqrt{2}} \right| = 5\sqrt{2} \Rightarrow |7 + k| = 10 \Rightarrow 7 + k = \pm 10. \therefore k = 3 \text{ or } k = -17.$

OR

Part (b) (ii)

10 marks

Att 3

$y = x + k$
 $(x - 3)^2 + ((x + k) + 4)^2 = 50$
 $2x^2 + (2 + 2k)x + (8k + 25) = 0$
One point of contact $\Rightarrow (2 + 2k)^2 - 4 \cdot 2(k^2 + 8k - 25) = 0$
 $\Rightarrow k^2 + 14k - 51 = 0$
 $\Rightarrow (k - 3)(k + 17) = 0$
 $\Rightarrow k = 3, k = -17$

Blunders (-3)

- B1 Incorrect centre of circle
- B2 Error in perpendicular distance formula
- B3 Incorrect radius
- B4 One value of k only
- B5 Incorrect squaring
- B6 Error in factors

Slips (-1)

- S1 Arithmetic error

Attempts (3 marks)

- A1 Centre or radius correct
- A2 Some correct substitution into perpendicular formula
- A3 Some correct substitution of $y = x + k$ or equivalent into circle

Part (c)

20 (10, 5, 5) marks

Att (3, 2, 2)

(c) Two circles intersect at $p(2, 0)$ and $q(-2, 8)$. The distance from the centre of each circle to the common chord $[pq]$ is $\sqrt{20}$. Find the equations of the two circles.

Part (c) First equation in f and g
Equation in one variable
Finish

10 marks
5 marks
5 marks

Att 3
Att 2
Att 2

1 (c)

$$\text{Slope } pq = \frac{8-0}{-2-2} = -2 \Rightarrow \text{slope } st = \frac{1}{2}$$

$$\therefore \frac{4+f}{0+g} = \frac{1}{2} \Rightarrow g = 2f + 8.$$

$$|st|^2 = 20 \Rightarrow (0+g)^2 + (4+f)^2 = 20 \Rightarrow g^2 + f^2 + 8f = 4$$

$$\Rightarrow (2f+8)^2 + f^2 + 8f = 4 \Rightarrow 5f^2 + 40f + 60 = 0.$$

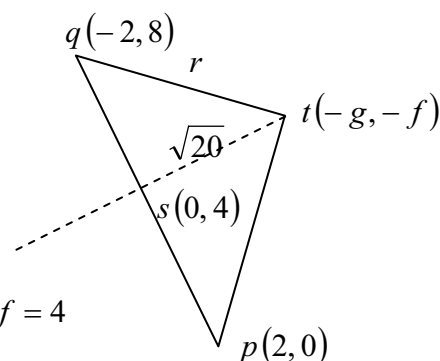
$$\therefore f^2 + 8f + 12 = 0 \Rightarrow (f+2)(f+6) = 0.$$

$$f = -2 \Rightarrow g = 4 \text{ or } f = -6 \Rightarrow g = -4.$$

$$\therefore \text{Centres are } (-4, 2) \text{ and } (4, 6), \quad r = \sqrt{40}.$$

$$\text{Circles are: } (x+4)^2 + (y-2)^2 = 40 \text{ and } (x-4)^2 + (y-6)^2 = 40.$$

$$\text{or } x^2 + y^2 + 8x - 4y - 20 = 0 \text{ and } x^2 + y^2 - 8x - 12y + 12 = 0$$



OR

Part (c) First equation in f and g
Equation in one variable
Finish

10 marks
5 marks
5 marks

Att 3
Att 2
Att 2

$$\text{Slope } pq = \frac{0-8}{2-(-2)} = -2$$

$$\text{Equation } pq: \quad y = -2(x-2) \text{ or } 2x + y - 4 = 0$$

$$\text{Perp. distance } (-g, -f) \text{ to } pq: \quad \left| \frac{-2g - f - 4}{\sqrt{5}} \right| = \sqrt{20}$$

$$\Rightarrow -2g - f - 4 = \pm 10 \Rightarrow 2g + f + 14 = 0 \text{ and } 2g + f - 6 = 0$$

$$\text{Distance from } (0,4) \text{ (= midpoint } pq) \text{ to } (-g, -f) \Rightarrow (0+g)^2 + (4+f)^2 = 20$$

$$\text{Solving between } g^2 + (4+f)^2 = 20 \text{ and } 2g + f - 6 = 0 \text{ gives } g = 4 \text{ and } f = -2$$

$$\text{Solving between } g^2 + (4+f)^2 = 20 \text{ and } 2g + f + 14 = 0 \text{ gives } g = -4 \text{ and } f = -6$$

$$\text{Eq. 1: } x^2 + y^2 + 8x - 4y + c = 0$$

$$(2, 0) \text{ on circle } \Rightarrow c = -20 \Rightarrow x^2 + y^2 + 8x - 4y - 20 = 0$$

$$\text{Eq2: Same method } \Rightarrow c = 12 \Rightarrow x^2 + y^2 - 8x - 12y + 12 = 0$$

OR

Part (c) First equation in f and g
Equation in one variable
Finish

10 marks
5 marks
5 marks

Att 3
Att 2
Att 2

1 (c)

$$(2,0) \in \text{Circle} \Rightarrow 2^2 + 0 + 2g(2) + 2f(0) + c = 0$$

$$\Rightarrow 4g + c = -4 \Rightarrow c = -4g - 4$$

$$(-2,8) \in \text{Circle} \Rightarrow -4g + 16f + c + 68 = 0$$

$$\Rightarrow -4g + 16f - 4g - 4 + 68 = 0 \Rightarrow g = 2(f + 4)$$

$$s(\text{midpoint}) = (0,4)$$

$$\text{But } \sqrt{g^2 + (4+f)^2} = \sqrt{20} \Rightarrow g^2 + (4+f)^2 = 20$$

$$\Rightarrow (2(f+4))^2 + (4+f)^2 = 20 \Rightarrow 5(f+4)^2 = 20$$

$$\Rightarrow (f+4)^2 = 4 \Rightarrow f+4 = \pm 2 \Rightarrow f = -6 \text{ and } -2$$

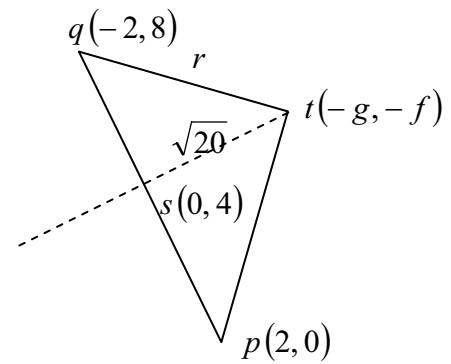
$$f = -6 \Rightarrow g = -4 \Rightarrow c = 12$$

$$f = -2 \Rightarrow g = 4 \Rightarrow c = -20$$

Circles :

$$x^2 + y^2 + 8x - 4y - 20 = 0$$

$$x^2 + y^2 - 8x - 12y + 12 = 0$$



OR

**Part (c) First equation in f and g
Equation in one variable
Finish**

**10marks
5 marks
5 marks**

**Att 3
Att 2
Att 2**

$$|pq| = \sqrt{(2+2)^2 + (0-8)^2} = \sqrt{80} \Rightarrow |ps| = \sqrt{20}$$

$$|pt|^2 = 20 + 20 = 40 \Rightarrow |pt| = \sqrt{40}$$

$\therefore p(2,0)$ as centre of a circle with radius $\sqrt{40}$

$$\Rightarrow (x-2)^2 + y^2 = 40$$

But $(-g, -f)$ on circle

$$\Rightarrow (-g-2)^2 + (0+f)^2 = 40$$

st is a chord.

$$\text{Slope } pq = \frac{8-2}{-2-2} = -2 \Rightarrow \text{slope } st = \frac{1}{2}$$

$$\Rightarrow \frac{4+f}{0+g} = \frac{1}{2} \Rightarrow g = 2f + 8$$

$$\therefore (-2f-8-2)^2 + f^2 = 40$$

$$\Rightarrow 5f^2 + 40f + 60 = 0 \Rightarrow f^2 + 8f + 12 = 0$$

$$\Rightarrow (f+6)(f+2) = 0 \Rightarrow f = -2 \text{ and } f = -6$$

$$f = -6 \Rightarrow g = -4 \Rightarrow c = 12$$

$$f = -2 \Rightarrow g = 4 \Rightarrow c = -20$$

Circles

$$x^2 + y^2 + 8x - 4y - 20 = 0$$

$$x^2 + y^2 - 8x - 12y + 12 = 0$$

Blunders (-3)

- B1 Error in distance formula
- B2 Error in mid point formula
- B3 Error in perpendicular distance formula
- B4 Incorrect application of Pythagoras formula
- B5 Error in slope formula
- B6 Error in squaring
- B7 Error in factors
- B8 Equation of one circle only

Slips (-1)

- S1 Arithmetic error

Attempts (3,2,2 marks)

- A1 Mid point or slope pq
- A2 c expressed in terms of g
- A3 Radius only

QUESTION 2

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (10, 10) marks	Att (3, 3)
Part (c)	20 (10, 10) marks	Att (3, 3)

Part (a) **10 (5, 5) marks** **Att (2, 2)**

2 (a)

(a) If $\vec{a} = 2\vec{i} + \vec{j}$, $\vec{b} = -\vec{i} + 5\vec{j}$, find the unit vector in the direction of \vec{ab} .

Part (a) \vec{ab} . **5 marks** **Att 2**
Finish **5 marks** **Att 2**

2 (a)

$$\vec{ab} = \vec{b} - \vec{a} = -\vec{i} + 5\vec{j} - 2\vec{i} - \vec{j} = -3\vec{i} + 4\vec{j}.$$

$$|\vec{ab}| = |-3\vec{i} + 4\vec{j}| = \sqrt{9 + 16} = 5.$$

$$\text{Unit vector} = \frac{\vec{ab}}{|\vec{ab}|} = \frac{-3\vec{i} + 4\vec{j}}{5} = -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.$$

Blunders (-3)

- B1 Error in $\vec{ab} = \vec{b} - \vec{a}$
- B2 Error in formula for norm of vector
- B3 Answer not expressed in correct form

Slips (-1)

- S1 Arithmetic error

Attempts (2,2 marks)

- A1 Norm formula with some substitution
- A2 $\vec{ab} = \vec{b} - \vec{a}$ and stops

Part (b) **20 (10, 10) marks** **Att (3, 3)**

(b) In the triangle abc , p is a point on the side $[bc]$.

The point q lies outside the triangle such that $\vec{pq} = \vec{pb} + \vec{pc} - \vec{pa}$.

(i) Express \vec{q} in terms of \vec{a} , \vec{b} and \vec{c} .

(ii) Hence show that $abqc$ is a parallelogram.

(b) (i)

10 marks

Att 3

2 (b) (i)

$$\vec{pq} = \vec{pb} + \vec{pc} - \vec{pa} \Rightarrow \vec{q} - \vec{p} = \vec{b} - \vec{p} + \vec{c} - \vec{p} - \vec{a} + \vec{p}.$$

$$\therefore \vec{q} = \vec{b} + \vec{c} - \vec{a}.$$

Blunders (-3)

B1 \vec{pq} or equivalent expressed incorrectly

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 \vec{pq} or equivalent expressed correctly

(b) (ii)

10 marks

Att 3

2 (b) (ii)

$$\text{By part (i): } \vec{q} = \vec{b} + \vec{c} - \vec{a} \Rightarrow \vec{q} - \vec{b} = \vec{c} - \vec{a} \Rightarrow \vec{bq} = \vec{ac}.$$

$\therefore abqc$ is a parallelogram.

Blunders (-3)

B1 $\vec{c} - \vec{a} \neq \vec{ac}$

B2 $\vec{q} - \vec{b} \neq \vec{bq}$

B3 No conclusion or incorrect conclusion

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 $\vec{q} - \vec{b} = \vec{c} - \vec{a}$

Part (c)

20 (10, 10) marks

Att (3, 3)

(c) (i) $\vec{p} = 12\vec{i} + 5\vec{j}$ and $\vec{q} = 3\vec{i} + 4\vec{j}$.

Find the value of the scalar k such that

$$k \left| \vec{p}^\perp - \vec{q} \right| = \left| \vec{p}^\perp \right| - \left| \vec{q} \right|.$$

(ii) Prove that for all vectors \vec{r} and \vec{s}

$$\left(\vec{r} - \vec{s} \right)^\perp = \vec{r}^\perp - \vec{s}^\perp.$$

Part (c) (i)**10 marks****Att 3****2 (c) (i)**

$$k \left| \vec{p}^\perp - \vec{q} \right| = \left| \vec{p}^\perp \right| - \left| \vec{q} \right| \Rightarrow k \left| -5\vec{i} + 12\vec{j} - 3\vec{i} - 4\vec{j} \right| = \left| -5\vec{i} + 12\vec{j} \right| - \left| 3\vec{i} + 4\vec{j} \right|.$$

$$\therefore k \left| -8\vec{i} + 8\vec{j} \right| = 13 - 5 \Rightarrow \sqrt{128}k = 8 \Rightarrow 8\sqrt{2}k = 8 \Rightarrow k = \frac{1}{\sqrt{2}} \Rightarrow k = \frac{\sqrt{2}}{2}.$$

Blunders (-3)

- B1 \vec{p}^\perp incorrect
 B2 Error in formula for norm of vector
 B3 k not in surd form

Slips (-1)

- S1 Arithmetic error

Attempts (3 marks)

- A1 Norm of \vec{q}
 A2 \vec{p}^\perp only

Part (c) (ii)**10 marks****Att 3****2 (c) (ii)**

$$\text{Let } \vec{r} = a\vec{i} + b\vec{j} \text{ and } \vec{s} = c\vec{i} + d\vec{j}. \therefore \vec{r} - \vec{s} = (a-c)\vec{i} + (b-d)\vec{j}.$$

$$\left(\vec{r} - \vec{s} \right)^\perp = -(b-d)\vec{i} + (a-c)\vec{j}$$

$$\vec{r}^\perp - \vec{s}^\perp = -b\vec{i} + a\vec{j} - \left(-d\vec{i} + c\vec{j} \right) = -(b-d)\vec{i} + (a-c)\vec{j} = \left(\vec{r} - \vec{s} \right)^\perp.$$

Blunders (-3)

- B1 \vec{r}^\perp incorrect
 B2 No conclusion or incorrect conclusion

Slips (-1)

- S1 Arithmetic error

Attempts (3 marks)

- A1 One related perpendicular correct
 A2 $\vec{r} - \vec{s}$ expressed in terms of \vec{i} and \vec{j}
 A3 Numerical values for \vec{r} and \vec{s} fully worked out 'correctly'.

QUESTION 3

Part (a)	10 marks	Att 3
Part (b)	20 (10, 10) marks	Att (3, 3)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 marks** **Att 3**

3 (a)

Find the equation of the line that contains the point $(1, 0)$ and passes through the point of intersection of the lines $2x - y + 6 = 0$ and $10x + 3y - 2 = 0$.

Part (a) **10 marks** **Att 3**

3 (a)

$$6x - 3y + 18 = 0$$

$$10x + 3y - 2 = 0$$

$$\frac{\quad}{16x + 16 = 0} \Rightarrow x = -1 \text{ and } y = 4.$$

$$(1, 0) \text{ and } (-1, 4) \Rightarrow m = \frac{0 - 4}{1 + 1} = -2.$$

$$\therefore \text{Equation of line : } y - 0 = -2(x - 1) \Rightarrow 2x + y - 2 = 0.$$

Blunders (-3)

B1 Error in slope formula

B2 Error in equation of line formula

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 One co-ordinate of point of intersection

A2 $2x - y + 6 + \lambda(10x + 3y - 2) = 0$

Part (b) **20 (10, 10) marks** **Att (3, 3)**

(b) (i) Prove that the measure of one of the angles between two lines with slopes m_1 and m_2 is given by

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

(ii) Find the equations of the two lines that pass through the point $(6, 1)$ and make an angle of 45° with the line $x + 2y = 0$.

3 (b) (i)

Slope $L_1 = m_1$ and slope $L_2 = m_2$.

Let θ_1 and θ_2 be the positive angles made by L_1 and L_2 respectively with the positive sense of the x -axis.

Then $\tan \theta_1 = m_1$ and $\tan \theta_2 = m_2$

Case 1: ($\theta_1 > \theta_2$)

$$\theta_1 = \theta + \theta_2 \Rightarrow \theta = \theta_1 - \theta_2.$$

$$\tan \theta = \tan(\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}$$

$$\therefore \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

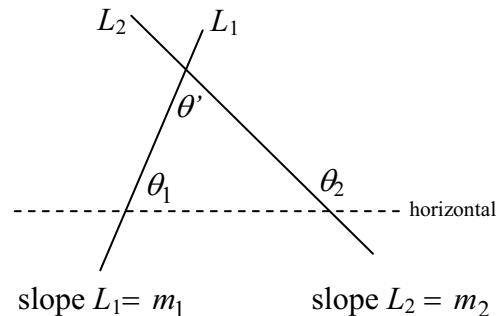
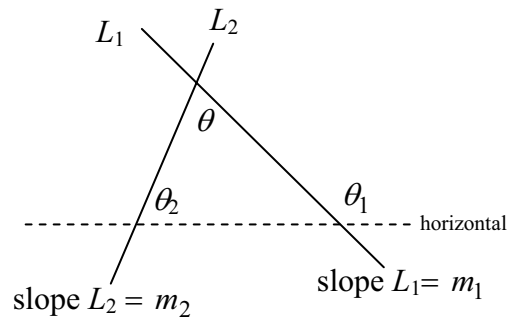
Case 2: ($\theta_1 < \theta_2$)

$$\theta_2 = \theta' + \theta_1 \Rightarrow \theta' = -(\theta_1 - \theta_2)$$

$$\tan \theta' = -\tan(\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}$$

$$= -\frac{m_1 - m_2}{1 + m_1 m_2}.$$

In this case, the other angle between the lines is $\theta = 180^\circ - \theta'$, giving $\tan \theta = -\tan \theta'$.



* One case to be accepted for full marks

Blunders (-3)

B1 Error in expressing θ in terms of θ_1 and θ_2

B2 Error in expansion of $\tan(\theta_1 - \theta_2)$

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 $\theta_1 = \theta + \theta_2$ and stops

Part (c)

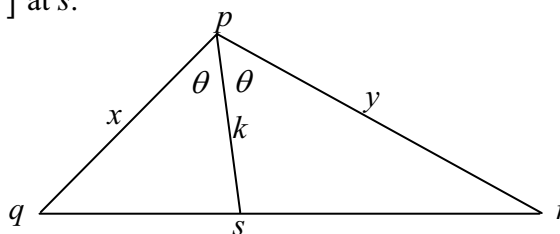
20 (10, 5, 5) marks

Att (3, 2, 2)

(c) The bisector of $\angle qpr$ meets $[qr]$ at s .

$$|\angle qpr| = 2\theta, |pq| = x,$$

$$|pr| = y \text{ and } |ps| = k.$$



(i) Find the area of the triangle pqs in terms of x , k and θ .

(ii) Show that $k = \frac{2xy \cos \theta}{x + y}$.

(c) (i)

10 marks

Att 3

4 (c) (i)

$$\text{Area triangle } pqs = \frac{1}{2} xk \sin \theta.$$

Blunders (-3)

B1 Area not in required form

B2 $\frac{1}{2}$ omitted in formula

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 Area = $\frac{1}{2}$ product of two sides \times sine of included angle, with some substitution

(c) (ii) Set up equation

5 marks

Att 2

Finish

5 marks

Att 2

4 (c) (ii)

$$\text{Area triangle } pqr = \text{area triangle } pqs + \text{area triangle } psr.$$

$$\therefore \frac{1}{2} xy \sin 2\theta = \frac{1}{2} xk \sin \theta + \frac{1}{2} ky \sin \theta.$$

$$\Rightarrow 2xy \sin \theta \cos \theta = k \sin \theta (x + y) \Rightarrow k = \frac{2xy \cos \theta}{x + y}.$$

Blunders (-3)

B1 $\sin 2\theta$ expanded incorrectly

B2 Error in factors

B3 k not in required form

B4 No conclusion or no conclusion

Slips (-1)

S1 Arithmetic error

Attempts (2, 2 marks)

A1 Area of triangle pqr

QUESTION 5

Part (a)	10 marks	Att 3
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 marks** **Att 3**

5 (a) Find all the solutions of the equation $\cos^2 x - \cos x = 0$, where $0^\circ \leq x \leq 180^\circ$.

5 (a) $\cos^2 x - \cos x = 0 \Rightarrow \cos x(\cos x - 1) = 0$

$\cos x = 0 \Rightarrow x = 90^\circ$ or $\cos x = 1 \Rightarrow x = 0^\circ$

Solution is $\{0^\circ, 90^\circ\}$

Blunders (-3)

- B1 Incorrect factors
- B2 Each incorrect value
- B3 Each omitted value or 'extra' value

Slips (-1)

- S1 Arithmetic error

Attempts (3 marks)

- A1 $\cos x = 0$ or $\cos x - 1 = 0$

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) The function $f : x \rightarrow \sin^{-1} x$ is defined for $-1 \leq x \leq 1$.

(i) Copy and complete the table of values of f below.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$f(x)$			$-\frac{\pi}{6}$				

(ii) Draw the graph of $y = f(x)$ on graph paper, noting that $\frac{\sqrt{3}}{2} \approx 0.87$.

Scale the y -axis in terms of π .

(iii) State, with reason, whether each of the following statements is true.

A: "If $\sin x_1 = \sin x_2$, then $x_1 = x_2$ ".

B: "If $\sin^{-1} x_1 = \sin^{-1} x_2$, then $x_1 = x_2$ ".

(b) (i)

5 marks

Att 2

5 (b) (i)

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$f(x)$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Slips (-1)

S1 Each incorrect entry to max of 3

Attempts (2 marks)

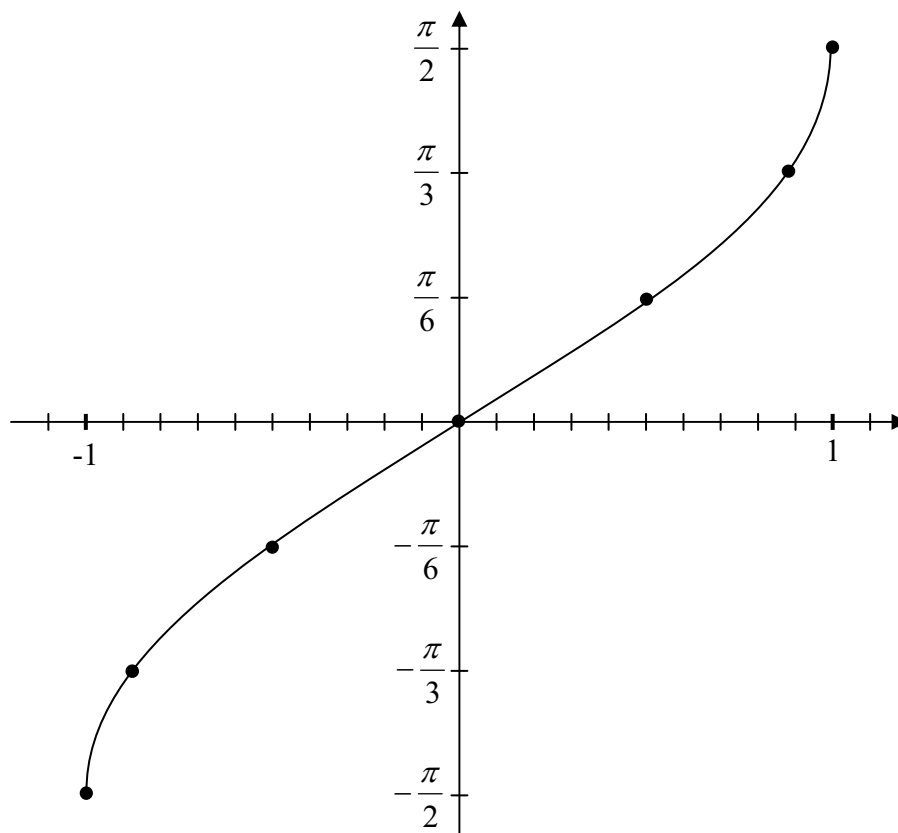
A1 One correct entry

(b) (ii)

5 marks

Att 2

5 (b) (ii)



Blunders (-3)

B1 x axis scaled in terms of π (instead of y axis)

B2 Error in scales

B3 Not joining points

Slips (-1)

S1 Each incorrect plot to max of 3.

Attempts (2 marks)

A1 Axes with some correct scale

A2 One point correctly indicated

(b) (iii) A
B

5 marks
5 marks

Att 2
Att 2

5 (b) (iii)

A is False:

For example, $\sin 150^\circ = \sin 30^\circ$, while $150^\circ \neq 30^\circ$

or

A horizontal line can cut the graph of $y = \sin(x)$ more than once.

B is True:

A horizontal line can't cut the graph of $y = \sin^{-1} x$ more than once.

or

\sin^{-1} is strictly increasing on its domain

or

\sin^{-1} is bijective

Blunders (-3)

B1 Correct answer no reason given

B2 Correct answer, incorrect reason

Slips (-1)

S1 Arithmetic error

Attempts (2, 2 marks)

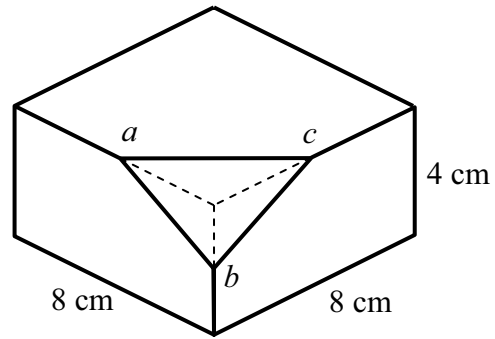
A1 $\sin 150^\circ = \sin 30^\circ$ or equivalent

Part (c)**20 (5, 5, 5, 5) marks****Att (2, 2, 2, 2)**

(c) A rectangular block of cheese measures
8 cm × 8 cm × 4 cm.

One corner is cut away from the block,
in such a way that three of the edges are
cut through their midpoints a , b and c .

Find the area of the triangular face abc
created by the cut.

**(c) $|ab|$ or $|bc|$** **5 marks****Att 2** **$|ac|$** **5 marks****Att 2****Cos****5 marks****Att 2****Finish****5 marks****Att 2****5 (c)**

$$|ab|^2 = 4^2 + 2^2 \Rightarrow |ab| = |bc| = \sqrt{20}.$$

$$|ac|^2 = 4^2 + 4^2 \Rightarrow |ac| = \sqrt{32}.$$

$$\cos \angle abc = \frac{|ab|^2 + |bc|^2 - |ac|^2}{2|ab||bc|} = \frac{20 + 20 - 32}{40} = \frac{8}{40} = \frac{1}{5}.$$

$$\therefore \sin \angle abc = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5}$$

$$\Rightarrow \text{area triangle } abc = \frac{1}{2}|ab||bc|\sin \angle abc = \frac{1}{2}(\sqrt{20})(\sqrt{20})\frac{2\sqrt{6}}{5} = 4\sqrt{6} \text{ cm}^2.$$

OR**(c) $|ab|$ or $|bc|$** **5 marks****Att 2** **$|ac|$** **5 marks****Att 2** **h** **5 marks****Att 2****Finish****5 marks****Att 2**

abc is an isosceles triangle.

Taking $|ac| = \sqrt{32}$ as base, let h be perpendicular height .

$$\therefore (\sqrt{20})^2 = h^2 + \left(\frac{1}{2}\sqrt{32}\right)^2 \Rightarrow h^2 = 12 \Rightarrow h = 2\sqrt{3}$$

$$\text{Area } \Delta abc = \frac{1}{2}\sqrt{32} \cdot 2\sqrt{3} = \sqrt{96} = 4\sqrt{6} \text{ cm}^2$$

Blunders (-3)

B1 Pythagoras incorrect

B2 Incorrect substitution into cosine formula

B3 Incorrect area formula

B4 Area not calculated

B5 $\sin A$ incorrectly evaluated from $\cos A$

Slips (-1)

S1 Arithmetic error

S2 Units omitted

Attempts (2, 2, 2, 2 marks)

A1 Incorrect use of Pythagoras

A2 Some substitution into cosine formula

QUESTION 6

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 10) marks	Att (2, 2, 3)

Part (a) **10 (5, 5) marks** **Att (2, 2)**

- (a) A student taking a literature course has to read three novels from a list of ten novels.
- (i) How many different selections of three novels are possible?
- (ii) Two of the ten novels are by the same author. How many selections are possible if the student wishes to choose three novels by different authors?

(a) (i) **5 marks** **Att 2**

6 (a) (i) Number of selections ${}^{10}C_3 = 120$

Blunders (-3)

B1 $10 \times 9 \times 8$

Attempts(2 marks)

A1 ${}^{10}C_x, x \in \mathbf{N}$

(a) (ii) **5 marks** **Att 2**

6 (a) (ii) Number of selections $= {}^2C_1 \times {}^8C_2 + {}^8C_3 = 56 + 56 = 112.$

or ${}^{10}C_3 - {}^8C_1 = 120 - 8 = 112$

or ${}^9C_3 + {}^8C_2 = 84 + 28 = 112$

Blunders (-3)

B1 2C_1 or equivalent missing

B2 ${}^2C_1 \times {}^8C_2 \times {}^8C_3$

B3 ${}^2C_1 \times {}^8C_2$

B4 8C_1 or 9C_3

Attempts (2 marks)

A1 8C_2 or 8C_3 or 8C_1

Part (b)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

- (b) (i) In how many different ways can eight people be seated in a row?
- (ii) Three girls and five boys sit in a row, arranged at random.
Find the probability that the three girls are seated together.
- (iii) Three girls and n boys sit in a row, arranged at random.
If the probability that the three girls are seated together is $\frac{1}{35}$,
find the value of n .

Part (b) (i)

5 marks

Att 2

6 (b) (i)

Number of ways = $8!$.

Blunders (-3)

B1 8C_8 or 8^8

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$

Part (b) (ii)

5 marks

Att 2

6 (b) (ii)

Number of possible arrangements = $8!$.

Number of favourable arrangements = $6! \times 3!$.

$$\text{Probability} = \frac{6! \times 3!}{8!} = \frac{6}{56} = \frac{3}{28}.$$

Blunders (-3)

B1 Incorrect number of possible outcomes

B2 Incorrect number of favourable outcomes (e.g. $5! \cdot 3!$)

B3 $6! + 3!$

B4 $6! \times 3$

B5 No divisor

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 Correct number of favourable outcomes

A2 Correct number of possible outcomes

Part (b) (iii) Probability
Finish

5 marks
5 marks

Att 2
Att 2

6 (b) (iii)

Number of possible arrangements = $(n + 3)!$.

Number of favourable arrangements = $(n + 1)! \times 3!$.

$$\therefore \text{Probability} = \frac{(n + 1)! \times 3!}{(n + 3)!} = \frac{6}{(n + 3)(n + 2)}.$$

$$\therefore \frac{6}{(n + 3)(n + 2)} = \frac{1}{35}.$$

$$\therefore n^2 + 5n + 6 = 210 \Rightarrow n^2 + 5n - 204 = 0.$$

$$(n - 12)(n + 17) = 0 \Rightarrow n = 12, \text{ as } n \neq -17.$$

Solution is $n = 12$.

Blunders (-3)

- B1 Incorrect number of possible outcomes
- B2 Incorrect number of favourable outcomes
- B3 Error in simplifying factorials
- B4 Error in factors of quadratic equation

Slips (-1)

- S1 Arithmetic error
- S2 $n = -17$ not excluded

Attempts (2,2 marks)

- A1 Correct number of possible outcomes
- A2 Correct number of favourable outcomes
- A3 Use of $n + 1$

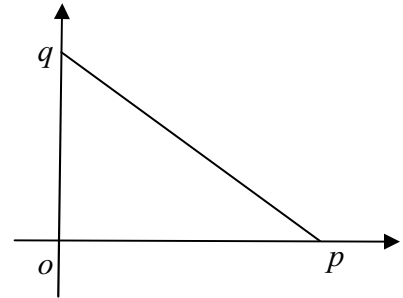
Part (c)

20 (5, 5, 10) marks

Att (2, 2, 3)

(c) x and y are randomly selected integers with $1 \leq x \leq 10$ and $1 \leq y \leq 10$.
 p is the point with coordinates $(x, 0)$ and q is the point with coordinates $(0, y)$.
 Find the probability that

- (i) the slope of pq is equal to -1
- (ii) the slope of pq is greater than -1
- (iii) the length of $[pq]$ is less than or equal to 5.



- (c) (i)
- (ii)
- (iii)

5 marks
5 marks
10 marks

Att 2
Att 2
Att 3

		x										
		1	2	3	4	5	6	7	8	9	10	
(i) $\bigcirc = \frac{10}{100} = \frac{1}{10}$ (ii) $\blacksquare = \frac{45}{100} = \frac{9}{20}$ (iii) $\checkmark = \frac{15}{100} = \frac{3}{20}$	y	1	$\bigcirc\checkmark$	$\blacksquare\checkmark$	$\blacksquare\checkmark$	$\blacksquare\checkmark$	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare
	2	\checkmark	$\bigcirc\checkmark$	$\blacksquare\checkmark$	$\blacksquare\checkmark$	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare
	3	\checkmark	\checkmark	$\bigcirc\checkmark$	$\blacksquare\checkmark$	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare
	4	\checkmark	\checkmark	\checkmark	\bigcirc	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare
	5					\bigcirc	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare
	6						\bigcirc	\blacksquare	\blacksquare	\blacksquare	\blacksquare	\blacksquare
	7							\bigcirc	\blacksquare	\blacksquare	\blacksquare	\blacksquare
	8								\bigcirc	\blacksquare	\blacksquare	\blacksquare
	9									\bigcirc	\blacksquare	\blacksquare
	10											\bigcirc

OR

Part (c) (i)

5 marks

Att 2

6 (c) (i)

Slope $pq = \frac{y-0}{0-x} = -\frac{y}{x}$ and $-\frac{y}{x} = -1$, when $x = y$.

\therefore Number of favourable outcomes is 10.

Number of possible outcomes is $10 \times 10 = 100$.

\therefore Probability = $\frac{10}{100} = \frac{1}{10}$.

Blunders (-3)

B1 $y \neq x$ implied

B2 Incorrect number of possible outcomes

B3 Incorrect number of favourable outcomes

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

- A1 Listing some favourable outcomes
- A2 Listing total number of outcomes

Part (c) (ii)

5 marks

Att 2

6 (c) (ii)

$$pq = \frac{y-0}{0-x} = -\frac{y}{x}, \quad -\frac{y}{x} > -1 \Rightarrow \frac{y}{x} < 1 \Rightarrow y < x.$$

\therefore Number of favourable outcomes is $9+8+7+6+5+4+3+2+1 = 45$.

Number of possible outcomes is $10 \times 10 = 100$.

$$\therefore \text{Probability} = \frac{45}{100} = \frac{9}{20}.$$

Blunders (-3)

- B1 $y < x$ not implied
- B2 Incorrect number of possible outcomes
- B3 Incorrect number of favourable outcomes

Slips (-1)

- S1 Arithmetic error

Attempts (2 marks)

- A1 Listing favourable outcomes
- A2 Listing total number of outcomes
- A3 Drawing a grid with some relevant items

Part (c) (iii)

10 marks

Att 3

6 (c) (iii)

$$|pq| = \sqrt{x^2 + y^2}, \quad |pq| \leq 5 \Rightarrow x^2 + y^2 \leq 25.$$

\therefore Favourable outcomes are

$x \in \{1, 2, 3, 4\}$ and $y \in \{1, 2, 3, 4\}$ but with $x = 4$ and $y = 4$ not included.

\therefore Number of favourable outcomes is $(4 \times 4) - 1 = 15$.

Number of possible outcomes is $10 \times 10 = 100$.

$$\therefore \text{Probability} = \frac{15}{100} = \frac{3}{20}.$$

Blunders (-3)

- B1 $x^2 + y^2 \leq 25$ or equivalent not implied
- B2 Incorrect number of possible outcomes
- B3 Incorrect number of favourable outcomes

Slips (-1)

- S1 Arithmetic error

Attempts (3 marks)

- A1 Listing some favourable outcomes
- A2 Listing total number of outcomes
- A3 Some use of Pythagoras

QUESTION 7

Part (a)	10 marks	Att 3
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 marks** **Att 3**

7 (a) The prices of four food items in a shopping basket are €3, €5, €1 and €6. Find the weighted mean price of these items using the weights 2, 3, 4 and 1 respectively.

Part (a) **10 marks** **Att 3**

7 (a)

$$\text{Weighted mean} = \frac{2(3) + 3(5) + 4(1) + 1(6)}{10} = \frac{31}{10} = €3.10.$$

Blunders (-3)

- B1 Sum of weights incorrect
- B2 Incorrect denominator
- B3 $x + w$ instead of xw for each term

Slips (-1)

- S1 Arithmetic error
- S2 Omission of currency symbol

Attempts (3 marks)

- A1 Sum of weights

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

- (b) (i) Solve the difference equation $u_{n+2} - 6u_{n+1} + 5u_n = 0$, where $n \geq 1$, given that $u_1 = 0$ and $u_2 = 20$.
- (ii) Find an expression in n for the sum of the terms $u_1 + u_2 + u_3 + \dots + u_n$.

Part (b) (i) Characteristic Equation **5 marks** **Att 2**

u_n **5 marks** **Att 2**

Finish **5 marks** **Att 2**

7 (b) (i)

$$u_{n+2} - 6u_{n+1} + 5u_n = 0 \Rightarrow x^2 - 6x + 5 = 0.$$

$$\therefore (x-1)(x-5) = 0 \Rightarrow x = 1 \text{ or } x = 5.$$

$$u_n = p(\alpha)^n + q(\beta)^n = p(1)^n + q(5)^n \Rightarrow u_n = p + q(5)^n.$$

$$u_1 = p + 5q = 0 \text{ and } u_2 = p + 25q = 20.$$

$$\therefore 20q = 20 \Rightarrow q = 1 \text{ and hence } p = -5.$$

$$\therefore u_n = -5 + 5^n.$$

Blunders (-3)

- B1 Error in setting up quadratic
- B2 Error in solving quadratic
- B3 Error in general term
- B4 Error in finding p and q

Slips (-1)

- S1 Arithmetic error

Attempts (2, 2,2 marks)

- A1 Substitution into quadratic formula
- A2 Attempt at finding p or q

Part (b) (ii)

5 marks

Att 2

7 (b) (ii)

$$\begin{aligned}u_1 + u_2 + u_3 + \dots + u_n &= \sum_{n=1}^n u_n = -5n + \sum_{n=1}^n 5^n \\ &= -5n + \frac{5(5^n - 1)}{5 - 1} = -5n + \frac{5}{4}(5^n - 1)\end{aligned}$$

Blunders (-3)

- B1 Error in forming geometric series
- B2 Error in sum of geometric series
- B3 Mishandling -5

Slips (-1)

- S1 Arithmetic error

Attempts (2 marks)

- A1 Using formula for sum to infinity of G.P.
- A2 Correct formula with some substitution
- A3 Listing at least 3 consecutive terms correctly

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) The two numbers a and b have mean \bar{x} and standard deviation σ_1 .
The three numbers c, d and e have mean \bar{x} and standard deviation σ_2 .
Find the standard deviation of the five numbers a, b, c, d and e in terms of σ_1 and σ_2 .

Part (c) Expressions for σ_1 and σ_2 5 marks Att 2
Both \bar{x} for $(a, b$ and $c, d, e)$ 5 marks Att 2
Mean a, b, c, d, e 5 marks Att 2
Finish 5 marks Att 2

7 (c)

$$\frac{a+b}{2} = \bar{x}, \quad \frac{c+d+e}{3} = \bar{x}.$$

$$\sigma_1 = \sqrt{\frac{(a-\bar{x})^2 + (b-\bar{x})^2}{2}}, \quad \sigma_2 = \sqrt{\frac{(c-\bar{x})^2 + (d-\bar{x})^2 + (e-\bar{x})^2}{3}}.$$

$$\text{Mean of } a, b, c, d, e = \frac{a+b+c+d+e}{5} = \frac{2\bar{x}+3\bar{x}}{5} = \bar{x}.$$

$$\begin{aligned} \text{Standard deviation of } a, b, c, d, e &= \sqrt{\frac{(a-\bar{x})^2 + (b-\bar{x})^2 + (c-\bar{x})^2 + (d-\bar{x})^2 + (e-\bar{x})^2}{5}} \\ &= \sqrt{\frac{2\sigma_1^2 + 3\sigma_2^2}{5}}. \end{aligned}$$

OR

Part (c) Expressions for σ_1 and σ_2 5 marks Att 2
Both \bar{x} (for a, b and $c, d, e)$ 5 marks Att 2
Mean a, b, c, d, e 5 marks Att 2
Finish 5 marks Att 2

$$\sum \frac{x^2}{n} - (\bar{x})^2 \Rightarrow \sigma_1^2 = \frac{a^2 + b^2}{2} - \bar{x}^2 \quad \text{and} \quad \sigma_2^2 = \frac{c^2 + d^2 + e^2}{3} - \bar{x}^2$$

$$\text{But } \frac{a+b}{2} = \bar{x} \quad \text{and} \quad \frac{c+d+e}{3} = \bar{x} \Rightarrow \frac{a+b+c+d+e}{5} = \frac{2\bar{x}+3\bar{x}}{5} = \bar{x}$$

$$\begin{aligned} \sigma^2 &= \frac{a^2 + b^2 + c^2 + d^2 + e^2}{5} - \bar{x}^2 \\ &= \frac{a^2 + b^2 + c^2 + d^2 + e^2 - 5\bar{x}^2}{5} \\ &= \frac{a^2 + b^2 - 2\bar{x}^2 + c^2 + d^2 + e^2 - 3\bar{x}^2}{5} \\ &= \frac{2\sigma_1^2 + 3\sigma_2^2}{5} \end{aligned}$$

$$\therefore \sigma = \sqrt{\frac{2\sigma_1^2 + 3\sigma_2^2}{5}}$$

Blunders (-3)

B1 Error in mean

B2 Error in standard deviation

Slips (-1)

S1 Arithmetic error

Attempts (2, 2, 2, 2 marks)

A1 Correct mean of a and b

A2 One correct standard deviation

A3 Expression for mean of a, b, c, d, e

QUESTION 8

Part (a)	15 (5, 5, 5) marks	Att (2, 2, 2)
Part (b)	20 (15, 5) marks	Att (5, 2)
Part (c)	15 (5, 5, 5) marks	Att (2, 2, 2)

Part (a) 15 (5, 5, 5) marks Att (2, 2, 2)

8. (a) Use integration by parts to find $\int xe^{4x} dx$.

Part (a) Assign parts	5 marks	Att 2
$\frac{du}{dx}$ and v	5 marks	Att 2
Finish	5 marks	Att 2

8 (a)

$$\int xe^{4x} dx = uv - \int v du, \text{ where } u = x \Rightarrow du = dx \text{ and } dv = e^{4x} dx \Rightarrow v = \frac{1}{4}e^{4x}.$$

$$\therefore \int xe^{4x} dx = \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x} dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + c = \frac{e^{4x}}{16}(4x - 1) + c.$$

Blunders (-3)

- B1 Incorrect differentiation or integration
- B2 Incorrect 'parts' formula

Slips (-1)

- S1 Arithmetic error
- S2 Omits constant of integration

Attempts (2,2,2 marks)

- A1 One correct assigning to parts formula
- A2 Correct differentiation or integration

Part (b)

20 (15, 5) marks

Att (5, 2)

(b) (i) Derive the first four terms of the Maclaurin series for $f(x) = \sqrt{1+x}$.

(ii) Given that this series converges for $-1 < x < 1$, use these four terms to find an approximation for $\sqrt{17}$, as a fraction.

(b) (i)

15 marks

Att 5

8 (b) (i)

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$$
$$f(x) = \sqrt{1+x} \Rightarrow f(0) = 1.$$
$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}.$$
$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}.$$
$$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}.$$
$$\therefore f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

Blunders (-3)

B1 Incorrect differentiation

B2 Incorrect evaluation of $f^{(n)}(0)$

B3 Each term not derived

B4 Error in Maclaurin series

Slips(-1)

S1 Arithmetic error

Attempts (5 marks)

A1 Correct expansion for $f(x)$ given but not derived

A2 $f(0)$ correct

A3 A correct differentiation

A4 Any one correct term

(b) (ii)

5 marks

Att 2

8 (b) (ii)

$$\sqrt{17} = \sqrt{16+1} = 4\sqrt{1+\frac{1}{16}} = 4\sqrt{1+x}, \text{ for } x = \frac{1}{16}.$$

$$\therefore \sqrt{17} = 4\left[1 + \frac{1}{32} - \frac{1}{2048} + \frac{1}{65536} + \dots\right] = 4\left[\frac{67553}{65536}\right] = \frac{67553}{16384}.$$

Blunders (-3)

B1 Mishandling of $\sqrt{16+1}$

B2 Answer not in form $\frac{a}{b}$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 17 as sum of 16 and 1 or 17 as sum of 9 and 8

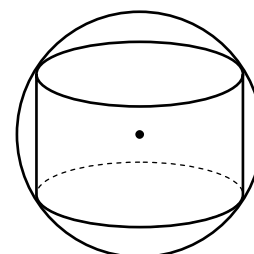
A2 Answer in decimal form with relevant work

Part (c)

15 (5, 5, 5) marks

Att (2, 2, 2)

(c) The diagram shows a cylinder inscribed in a sphere.
 The cylinder has height $2x$ and radius r .
 The sphere has fixed radius a .



(i) Express r in terms of a and x .

(ii) Find, in terms of a , the maximum possible volume of the cylinder.

(c) (i)

5 marks

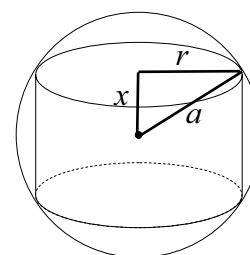
Att 2

8 (c) (i)

$$r^2 + x^2 = a^2$$

$$r^2 = a^2 - x^2$$

$$\Rightarrow r = \sqrt{a^2 - x^2}$$



Blunders (-3)

B1 Error in Pythagoras

B2 Incorrect side in triangle

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 $a^2 = r^2 + x^2$

(c)(ii) Volume in terms of x
Finish

5 marks
5 marks

Att 2
Att 2

8 (c) (ii)

Volume of cylinder = $V = \pi r^2 h$.

$$\therefore V = \pi(a^2 - x^2)2x = 2\pi a^2 x - 2\pi x^3.$$

$$\frac{dV}{dh} = 2\pi a^2 - 6\pi x^2 = 0 \text{ for max or min. } \Rightarrow x = \frac{a}{\sqrt{3}}.$$

$$\frac{d^2V}{dh^2} = -12\pi x < 0, \text{ for } x = \frac{a}{\sqrt{3}}$$

$$\Rightarrow \text{maximum volume at } x = \frac{a}{\sqrt{3}}.$$

$$\therefore V = \pi \left(a^2 - \frac{a^2}{3} \right) \frac{2a}{\sqrt{3}} = \frac{4\pi a^3}{3\sqrt{3}} = \frac{4\sqrt{3}\pi a^3}{9}.$$

* $\frac{d^2V}{dh^2}$ not required

Blunders (-3)

B1 Error in differentiation

B2 Error in finding x

B3 Error in indices

Slips (-1)

S1 Arithmetic error

Attempts (2, 2 marks)

A1 Some part of correct substitution into volume

A2 Some correct differentiation

A3 $\frac{dV}{dx} = 0$ indicated for max or min

QUESTION 9

Part (a)	10 (5, 5) marks	Att (2, 2)
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 marks** **Att 3**

9. (a) A and B are independent events such that $P(A) = 0.25$ and $P(A \cup B) = 0.55$.
Find $P(B)$.

Part (a) Apply independence rule **5 marks** **Att 2**
Finish **5 marks** **Att 2**

9 (a)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

$$\therefore 0.55 = 0.25 + P(B) - (0.25)P(B) \Rightarrow 0.75P(B) = 0.3$$

$$\therefore P(B) = 0.4$$

OR

A and B independent $\Leftrightarrow A'$ and B' independent.
 $P(A')P(B') = P(A' \cap B') = P((A \cup B)')$
 $0.75P(B') = 0.45$
 $P(B') = 0.6$
 $P(B) = 0.4$

OR

$$P(A \cap B) = P(A)P(B)$$

$$x = (0.25)(0.3 + x)$$

$$= 0.075 + 0.25x$$

$$0.75x = 0.075$$

$$x = 0.1$$

$$P(B) = 0.4$$

The Venn diagram shows two overlapping circles, A and B, within a rectangular frame. Circle A is on the left and circle B is on the right. The intersection of A and B is labeled 'x'. The area of circle A that does not overlap with B is labeled '0.25 - x'. The area of circle B that does not overlap with A is labeled '0.3'. The total area of circle B is labeled '0.45' at the bottom right.

Blunders (-3)

- B1 $P(A \cap B) \neq P(A)P(B)$
- B2 $P(A \cup B) = P(A) + P(B) + P(A)P(B)$

Slips (-1)

- S1 Arithmetic error

Attempts (2,2 marks)

- A1 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Part (b)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

- (b) A person plays a game that involves throwing five hoops at a peg. The following table gives the probability distribution for the number of hoops that land on the peg.

x	0	1	2	3	4	5
$P(x)$	0.01	0.08	0.23	0.34	0.26	0.08

Find the mean and the standard deviation of the distribution.

Part (b) Mean

5 marks

Att 2

Deviations

5 marks

Att 2

$(x - \bar{x})^2 \cdot P(x)$ expressed

5 marks

Att 2

Finish

5 marks

Att 2

9 (b)

$$\text{Mean} = \bar{x} = \sum_{x=0}^5 xP(x) = 0.08 + 0.46 + 1.02 + 1.04 + 0.4 = 3.$$

$$\text{Standard deviation} = \sigma = \sqrt{\sum_{x=0}^5 (x - \bar{x})^2 \cdot P(x)}.$$

$$= \sqrt{(0-3)^2(0.01) + (1-3)^2(0.08) + (2-3)^2(0.23) + (3-3)^2(0.34) + (4-3)^2(0.26) + (5-3)^2(0.08)}$$

$$\therefore \sigma = \sqrt{0.09 + 0.32 + 0.23 + 0 + 0.26 + 0.32} = \sqrt{1.22}.$$

Blunders (-3)

B1 $\sum P(x)$ incorrect

B2 $\sum x$ denominator for mean

B3 Use of $\sum (x + P(x))$

B4 Mishandles deviation

B5 Incorrect standard deviation formula

Slips (-1)

S1 Arithmetic error

Attempts (2,2,2,2 marks)

A1 Any correct $x \cdot P(x)$

A2 Correct formula with some substitution

A3 Any correct deviation

Part (c)**20 (5, 5, 5, 5) marks****Att (2, 2, 2, 2)**

(c) A coin is slightly bent and is thought to favour heads. Accordingly, it is tossed 100 times to test the null hypothesis that it is fair against the alternative hypothesis that it favours heads. In this experiment, 55 heads are observed.

- (i) Show that this result is not significant at the 5% level.
- (ii) How many times would the coin have to be tossed in an experiment in order that an observation of 55% heads *would* be regarded as significant at the 5% level?

(c) (i) σ **5 marks****Att 2****Finish****5 marks****Att 2****9 (c) (i)** H_0 : coin is fair.

This is a one tailed test.

$$p = \frac{1}{2}, q = \frac{1}{2}, n = 100. \quad \bar{x} = np = 50, \sigma = \sqrt{npq} = 5.$$

$$z = \frac{x - \bar{x}}{\sigma} = \frac{55 - 50}{5} = 1.$$

$$z = 1 < 1.645.$$

\therefore The observed result is not significant at the 5% level.

Blunders (-3)

- B1 Incorrect value of p or of q
 B2 Incorrect formula for mean
 B3 Incorrect formula for standard deviation
 B4 Error in standard units
 B5 Two tailed test
 B6 Misreads tables
 B7 Incorrect conclusion

Slips (-1)

- S1 Arithmetic error

Attempts (2,2 marks)

- A1 Correct value for p or q
 A2 Correct formula for mean with some substitution
 A3 Correct formula for standard deviation with some substitution
 A4 Correct expression for standard units with some substitution

(c) (ii) Value of z
Finish

5 marks
5 marks

Att 2
Att 2

9 (c) (ii)

$$\bar{x} = np = \frac{n}{2}, \quad \sigma = \sqrt{npq} = \frac{\sqrt{n}}{2}.$$

$$z = \frac{\frac{55n}{100} - \frac{n}{2}}{\frac{\sqrt{n}}{2}} = \frac{\frac{n}{10}}{\frac{\sqrt{n}}{2}} = \frac{\sqrt{n}}{10}.$$

$$\frac{\sqrt{n}}{10} > 1.645 \Rightarrow \sqrt{n} > 16.45 \Rightarrow n > 270.6.$$

\therefore 271 times

Blunders (-3)

- B1 Incorrect value of p or of q
- B2 Incorrect formula for mean
- B3 Incorrect formula for standard deviation
- B4 Error in standard units
- B5 Mishandles 55%
- B6 Incorrect conclusion

Slips (-1)

- S1 Arithmetic error
- S2 Stops at $n > 270.6$

Attempts (2,2 marks)

- A1 Correct value for p or q
- A2 Some substitution for standard units

QUESTION 10

Part (a)	10 marks	Att 3
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 marks** **Att 3**

10. (a) If a is the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, find $a \circ a$.

10 (a)

$$a \circ a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Blunders (-3)

B1 Each incorrect element (max. of 2)

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 Permutation incomplete

A2 One element correct with another repeated

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) The set $\{1, 2, 4, 5, 7, 8\}$ is a group under multiplication modulo 9.

(i) Draw up a Cayley table for the group.

(ii) Find a generator of the group.

(iii) Hence, or otherwise, find a subgroup of order 2 and a subgroup of order 3.

(b) (i) **5 marks** **Att 2**

10 (b) (i)

× mod 9	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

Blunders (-3)

B1 Not closed

Slips (-1)

S1 Arithmetic error

S2 Each incorrect entry to max of 3

Attempts (2 marks)

A1 Incomplete table

(b) (ii)

5 marks

Att 2

10 (b) (ii)

5 is a generator. $5^1 = 5, 5^2 = 7, 5^3 = 8, 5^4 = 4, 5^5 = 2, 5^6 = 1.$

or 2 is also a generator $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 7, 2^5 = 5, 2^6 = 1$

Blunders (-3)

B1 Error in verifying generator

B2 Identity or other element not shown in terms of generator

Attempts (2marks)

A1 Generator identified but not demonstrated for any element

A2 Attempts to establish a generator

Part (b) (iii)

10 (5, 5) marks

Att (2, 2)

10 (b) (iii)

Subgroup of order 2 is $\{1, 8\}.$

Subgroup of order 3 is $\{1, 4, 7\}$

Blunders (-3)

B1 Incorrect element in subgroup

Slips (-1)

S1 Arithmetic error

Attempts (2, 2 marks)

A1 Identity only correct element

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) (G, \circ) and $(H, *)$ are two groups with identities e_G and e_H respectively.

If $\phi : G \rightarrow H$ is an isomorphism, prove that

(i) $\phi(e_G) = e_H$.

(ii) $\phi(x^{-1}) = [\phi(x)]^{-1}$, for all $x \in G$.

(c) (i) Establish correspondence

5 marks

Att 2

Finish

5 marks

Att 2

10 (c) (i)

Let $x \in G$.

$\phi(x \circ e_G) = \phi(x) * \phi(e_G)$, because of isomorphism.

$\phi(x) = \phi(x) * \phi(e_G)$, as $x \circ e_G = x$.

$\Rightarrow \phi(e_G)$ is the identity in $(H, *)$.

$\therefore \phi(e_G) = e_H$.

Blunders (-3)

B1 Error in setting up correspondence in operators

B2 No conclusion

Slips (-1)

S1 Arithmetic error

Attempts (2,2 marks)

A1 $x \circ e_G = x$

(c) (ii) Establish correspondence

5 marks

Att 2

Finish

5 marks

Att 2

10 (c) (ii)

$$\phi(x \circ x^{-1}) = \phi(x) * \phi(x^{-1})$$

$\therefore \phi(e_G) = \phi(x) * \phi(x^{-1}) \Rightarrow \phi(x)$ and $\phi(x^{-1})$ are inverses.

$$\therefore \phi(x^{-1}) = [\phi(x)]^{-1}.$$

Blunders (-3)

B1 $x \circ x^{-1} \neq e_G$ or equivalent

B2 Not indicating $\phi(x)$ and $\phi(x^{-1})$ are inverses

B3 No conclusion

Slips (-1)

S1 Arithmetic error

Attempts (2,2 marks)

A1 $x \circ x^{-1} = e$

QUESTION 11

Part (a)	10 marks	Att 3
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) 10 marks Att 3

(a) Find the equation of the ellipse with foci $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$ and with eccentricity $\frac{\sqrt{7}}{4}$.

11 (a)

$$e = \frac{\sqrt{7}}{4} \text{ and } ae = \sqrt{7} \Rightarrow a = 4.$$

$$b^2 = a^2(1 - e^2) = 16\left(1 - \frac{7}{16}\right) = 9.$$

$$\therefore \text{ellipse: } \frac{x^2}{16} + \frac{y^2}{9} = 1.$$

Blunders (-3)

B1 Formula error

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 $ae = \sqrt{7}$ and stops

Part (b)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(b)	A transformation f is a similarity transformation if there exists a fixed number k such that $ f(a)f(b) = k ab $, for all a and b .
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	Show that angle measure is invariant under a similarity transformation.
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Part (b) f defined

5 marks

Att 2

$\cos \angle \phi$

5 marks

Att 2

Use of k^2

5 marks

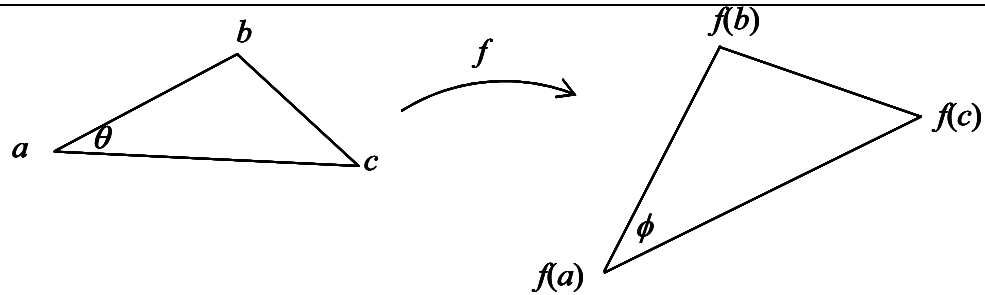
Att 2

Finish

5 marks

Att 2

11 (b)



The triangle abc is mapped onto the triangle $f(a)f(b)f(c)$ under a similarity transformation f . $\angle \theta$ is mapped onto $\angle \phi$.

$$\cos \angle \phi = \frac{|f(a)f(b)|^2 + |f(a)f(c)|^2 - |f(b)f(c)|^2}{2|f(a)f(b)||f(a)f(c)|} = \frac{k^2|ab|^2 + k^2|ac|^2 - k^2|bc|^2}{2k^2|ab||ac|}.$$

$$\therefore \cos \angle \phi = \frac{|ab|^2 + |ac|^2 - |bc|^2}{2|ab||ac|} = \cos \angle bac = \cos \angle \theta.$$

$\therefore |\angle \theta| = |\angle \phi|$, since both are in the range 0° to 180°

Hence angle measure is invariant under a similarity transformation.

Blunders (-3)

B1 Error in cosine formula

B2 Fails to identify $|f(a)f(b)| = k|ab|$ or equivalent

B3 No conclusion or incorrect conclusion

Slips (-1)

S1 Arithmetic error

Attempts (2, 2, 2, 2 marks)

A1 $\cos \theta$ with some substitution

Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c)	(i)	Define the term <i>conjugate diameters</i> of an ellipse.
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	(ii)	Prove that all parallelograms circumscribed to a given ellipse at the endpoints of conjugate diameters have the same area.
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(c) (i)

5 marks

Att 2

11 (c) (i)	If $ pq $ is a diameter of an ellipse E , then there is a second diameter $ rs $, such that $ pq $ bisects all chords of E on lines parallel to $ rs $ and vice versa. $ pq $ and $ rs $ are called <i>conjugate diameters</i> of the ellipse.
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Blunders (-3)

B1 Parallel property not indicated

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 Incomplete relevant diagram

Part (c) (ii) f

5 marks

Att 2

f^{-1}

5 marks

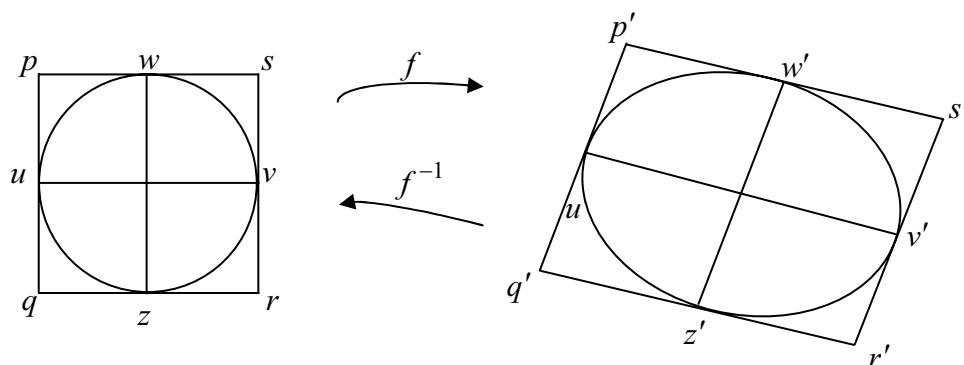
Att 2

Finish

5 marks

Att 2

11 (c) (ii)



$[u'v']$ and $[w'z']$ are conjugate diameters of ellipse E .

Tangents at their end points form the parallelogram $p'q'r's'$.

Under an affine transformation f^{-1} , the ellipse maps to the circle $x^2 + y^2 = 1$ and $p'q'r's'$ is mapped to $pqrs$.

$[uv]$ and $[wz]$ are conjugate diameters of the circle and $uv \perp wz$. The square $pqrs$ has fixed area 4 sq units.

\therefore Area $p'q'r's' = |\det(f)| \cdot \text{area } pqrs = 4 \det(f)$.

But $\det(f)$ is constant \Rightarrow area $p'q'r's'$ is constant.

\therefore Areas of all parallelograms at end points of conjugate diameters of an ellipse are equal.

Blunders (-3)

B1 Fails to identify conjugate diameters of circle are perpendicular

B2 Fails to identify $\det(f)$ is constant and / or area $pqrs$ is constant

Slips (-1)

S1 Arithmetic error

Attempts (2, 2, 2 marks)

A1 Some relevant mapping

MARCANNA BREISE AS UCHT FREAGAIRT TRÍ GHAEILGE

(Bonus marks for answering through Irish)

Ba chóir marcanna de réir an ghnáthráta a bhronnadh ar iarrthóirí nach ngnóthaíonn níos mó ná 75% d'iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónais sin a shlánú **síos**.

Déantar an cinneadh agus an ríomhaireacht faoin marc bónais i gcás gach páipéir ar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an ngnáthráta 5% i gcás iarrthóirí a ghnóthaíonn 225 marc nó níos lú, e.g. $198 \text{ marc} \times 5\% = 9.9 \Rightarrow \text{bónas} = 9 \text{ marc}$.

Má ghnóthaíonn an t-iarrthóir níos mó ná 225 marc, ríomhtar an bónas de réir na foirmle $[300 - \text{bunmharc}] \times 15\%$, agus an marc bónais sin a shlánú **síos**. In ionad an ríomhaireacht sin a dhéanamh, is féidir úsáid a bhaint as an tábla thíos.

Bunmharc	Marc Bónais
226	11
227 – 233	10
234 – 240	9
241 – 246	8
247 – 253	7
254 – 260	6
261 – 266	5
267 – 273	4
274 – 280	3
281 – 286	2
287 – 293	1
294 – 300	0

